

# QUANTIZATION OF Q-HAMILTONIAN $SU(2)$ -SPACES

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ABSTRACT. We will explain how to define the quantization of q-Hamiltonian  $SU(2)$ -spaces as push-forwards in twisted equivariant  $K$ -homology, and prove the ‘quantization commutes with reduction’ theorem for this setting. As applications, we show how the Verlinde formulas for flat  $SU(2)$  or  $SO(3)$ -bundles are obtained via localization in twisted  $K$ -homology.

*Dedicated to Hans Duistermaat on the occasion of his 65th birthday.*

## 1. INTRODUCTION

The theory of q-Hamiltonian  $G$ -spaces was introduced ten years ago in the paper *Lie group-valued moment maps* [1]. The motivation was to treat Hamiltonian loop group actions with proper moment maps in a purely finite-dimensional framework, obtaining for instance a finite-dimensional construction of the moduli space of flat  $G$ -bundles over a surface. Many of the standard constructions for ordinary Hamiltonian group actions on symplectic manifolds carried over to the new setting, but often with non-trivial ‘twists’. For example, all q-Hamiltonian  $G$ -spaces  $M$  carry a natural volume form [6], which may be viewed informally as a push-forward of the (ill-defined) Liouville form on the associated infinite-dimensional loop group space. This volume form admits an equivariant extension (but for a non-standard equivariant cohomology theory) [4], and the total volume may be computed by localization techniques, just as in the usual Duistermaat-Heckman theory [19].

One problem that had remained open until recently is how to define a ‘quantization’ of q-Hamiltonian spaces. In contrast to the Hamiltonian theory, the 2-form on a q-Hamiltonian space is usually degenerate. Hence, there is no obvious notion of a compatible almost complex structure, and the usual quantization as the equivariant index of a  $\text{Spin}_c$ -Dirac operator [17] is no longer possible. In a forthcoming paper [2], rather than trying to construct such an operator, we define the quantization more abstractly as the push-forward of a  $K$ -homology fundamental class  $[M]$ . This fundamental class is canonically defined as an element in *twisted* equivariant  $K$ -homology of  $M$ . Our construction defines a push-forward of this element to the twisted equivariant  $K$ -homology of a Lie group. The Freed-Hopkins-Teleman theorem [21, 20] identifies the latter with the fusion ring  $R_k(G)$  (Verlinde algebra), at an appropriate level  $k$ . We take the resulting element  $\mathcal{Q}(M) \in R_k(G)$  to be the ‘quantization’ of our q-Hamiltonian space. As in the usual Hamiltonian theory [23, 22, 31], the quantization procedure satisfies a ‘quantization commutes with reduction’ principle.

In the present paper, we will preview this quantization of  $q$ -Hamiltonian  $G$ -spaces for the simplest of simple compact Lie groups  $G = \mathrm{SU}(2)$ . Much of the general theory simplifies in this special case – for example, there is a fairly simple proof of the  $q$ -Hamiltonian ‘quantization commutes with reduction’ theorem. As an application, we explain, following [5], how the  $\mathrm{SU}(2)$ -Verlinde formulas are obtained in our theory. In the last Section, we will show how to derive Verlinde-type formulas for moduli spaces of flat  $\mathrm{SO}(3)$ -bundles. The paper will be largely self-contained, except for certain details that are better handled with the techniques from [2].

**Notation.** We fix the following notations and conventions for the Lie group  $\mathrm{SU}(2)$ . The group unit will be denoted  $e$ , and the non-trivial central element  $c = \mathrm{diag}(-1, -1)$ . We define an open cover by contractible subsets

$$(1) \quad \mathrm{SU}(2)_+ = \mathrm{SU}(2) \setminus \{c\}, \quad \mathrm{SU}(2)_- = \mathrm{SU}(2) \setminus \{e\}$$

with intersection the set  $\mathrm{SU}(2)_{\mathrm{reg}}$  of regular elements. We take the maximal torus  $T$  to consist of the diagonal matrices, isomorphic to  $\mathrm{U}(1)$  by the homomorphism

$$j: \mathrm{U}(1) \rightarrow T, \quad z \mapsto \mathrm{diag}(z, z^{-1}).$$

The Weyl group  $W = \mathbb{Z}_2$  acts on  $T$  by permutation of the diagonal entries, or equivalently on  $\mathrm{U}(1)$  by  $z \mapsto z^{-1}$ . We let  $\Lambda \subset \mathfrak{t}$  be the integral lattice (kernel of  $\exp|_{\mathfrak{t}}$ ) and  $\Lambda^* \subset \mathfrak{t}^*$  its dual, the (real) weight lattice. For any  $\mu \in \Lambda^*$  we denote by  $t \mapsto t^\mu$  the corresponding homomorphism  $T \rightarrow \mathrm{U}(1)$ ; the resulting 1-dimensional representation of  $T$  is denoted  $\mathbb{C}_\mu$ . The weight lattice is generated by the element  $\rho \in \Lambda^*$  such that  $\mathbb{C}_\rho$  is the defining representation of  $\mathrm{U}(1)$ . The corresponding positive root is  $\alpha = 2\rho$ . We will identify  $\mathfrak{su}(2)^* \cong \mathfrak{su}(2)$  using the *basic inner product*

$$\xi \cdot \xi' = \frac{1}{4\pi^2} \mathrm{tr}(\xi^\dagger \xi'), \quad \xi, \xi' \in \mathfrak{su}(2).$$

Similarly we identify  $\mathfrak{t} \cong \mathfrak{t}^*$  using the induced inner product. Under this identification,  $\Lambda = 2\Lambda^*$ , with generators  $\alpha = 2\pi i \mathrm{diag}(1, -1)$  and  $\rho = i\pi \mathrm{diag}(1, -1)$ .

For any subset  $A \subset \mathfrak{t}$ , we denote  $T_A = \exp(A) = \{\exp \xi \mid \xi \in A\}$ . Any conjugacy class in  $\mathrm{SU}(2)$  passes through a unique point in  $T_{[0, \rho]}$ , so that  $[0, \rho]$  labels the conjugacy classes. We will frequently use the equivariant diffeomorphism,

$$(2) \quad T_{(0, \rho)} \times \mathrm{SU}(2)/T \rightarrow \mathrm{SU}(2)_{\mathrm{reg}}, \quad (t, gT) \mapsto \mathrm{Ad}_g(t).$$

## 2. THE FUSION RING $R_k(\mathrm{SU}(2))$

In this Section, we review three simple descriptions of the level  $k$  fusion ring (Verlinde algebra)  $R_k(G)$  for the case  $G = \mathrm{SU}(2)$ . The fusion ring may be identified with the set of irreducible projective representations of the loop group  $L\mathrm{SU}(2)$  at level  $k$  [36], but we will not need that interpretation here.

**2.1. First description.** Let  $R(\mathrm{SU}(2))$  be the representation ring of  $\mathrm{SU}(2)$ , viewed as the ring of virtual characters. For  $m = 0, 1, 2, \dots$  let  $\chi_m \in R(\mathrm{SU}(2))$  be the character of the  $m+1$ -dimensional irreducible representation of  $\mathrm{SU}(2)$ . These form a basis of  $R(\mathrm{SU}(2))$  as a  $\mathbb{Z}$ -module, and the ring structure is given by

$$\chi_m \chi_{m'} = \chi_{m+m'} + \chi_{m+m'-2} + \dots + \chi_{|m-m'|}.$$

For  $k = 0, 1, 2, \dots$ , the *level  $k$  fusion ring* (or *Verlinde algebra*) is a quotient

$$R_k(\mathrm{SU}(2)) = R(\mathrm{SU}(2))/I_k(\mathrm{SU}(2))$$

by the ideal  $I_k(\mathrm{SU}(2))$  generated by the character  $\chi_{k+1}$ . Additively, the ideal is spanned by the characters  $\chi_{k+1}, \chi_{2k+3}, \chi_{3k+5}, \dots$ , together with all characters of the form  $\chi_{l'} - (-1)^r \chi_l$  where  $l \in \{0, \dots, k\}$ , and  $l'$  is obtained from  $l$  by  $r$  reflections across the set of elements  $k+1, 2k+3, 3k+5, \dots$ . It follows that as an Abelian group,  $R_k(\mathrm{SU}(2))$  is free with generators  $\tau_0, \dots, \tau_k$  the images of  $\chi_0, \dots, \chi_k$ . For example, if  $k = 4, m = 3, m' = 4$  we have

$$\chi_3 \chi_4 = \chi_1 + \chi_3 + \chi_5 + \chi_7 \Rightarrow \tau_3 \tau_4 = \tau_1 + \tau_3 + 0 - \tau_3 = \tau_1.$$

For any given level  $k$ , the element  $\tau_k \in R_k(\mathrm{SU}(2))$  defines an involution of the group  $R_k(\mathrm{SU}(2))$ ,

$$\tau_l \mapsto \tau_l \tau_k = \tau_{k-l}.$$

**2.2. Second description.** Let  $q$  be the  $2k+4$ -th root of unity,

$$q = e^{\frac{i\pi}{k+2}}.$$

Then  $I_k(\mathrm{SU}(2)) \subset R(\mathrm{SU}(2))$  may be described as the ideal of all characters vanishing at all points  $j(q^s)$ , for  $s = 1, \dots, k+1$ . Put differently, letting

$$T_{k+2} = \{t \in T \mid t^{2k+4} = e\}$$

be the cyclic subgroup generated by  $j(q)$ ,  $I_k(\mathrm{SU}(2))$  is the vanishing ideal of  $T_{k+2} \cap \mathrm{SU}(2)_{\mathrm{reg}} = T_{k+2}^{\mathrm{reg}}$ . Hence, for any  $t \in T_{k+2}^{\mathrm{reg}}$  the evaluation map  $\mathrm{ev}_t: R(\mathrm{SU}(2)) \rightarrow \mathbb{C}$  descends to an evaluation map

$$\mathrm{ev}_t: R_k(\mathrm{SU}(2)) \rightarrow \mathbb{C}, \quad \tau \mapsto \tau(t) = \mathrm{ev}_t(\tau).$$

For the basis elements one obtains, by the Weyl character formula,

$$\tau_l(j(q^s)) = \frac{q^{(l+1)s} - q^{-(l+1)s}}{q^s - q^{-s}}.$$

The orthogonality relations

$$(3) \quad \sum_{s=1}^{k+1} \frac{|q^s - q^{-s}|^2}{2k+4} \tau_l(j(q^s)) \tau_{l'}(j(q^s)) = \delta_{l,l'}$$

allow us to recover  $\tau \in R_k(\mathrm{SU}(2))$  from the values  $\tau(j(q^s))$  for  $s = 1, \dots, k$ . The coefficients in this sum may alternatively be written as

$$\frac{|q^s - q^{-s}|^2}{2k+4} = \left(\frac{k}{2} + 1\right)^{-1} \sin^2\left(\frac{\pi s}{k+2}\right).$$

**2.3. Third description.** The third way of describing the fusion ring is to write down the structure constants relative to the basis  $\tau_0, \dots, \tau_k$ . The *level  $k$  fusion coefficient*  $N_{l_1, l_2, l_3}^{(k)}$  for  $0 \leq l_i \leq k$  is the multiplicity of  $\tau_0$  in the triple product  $\tau_{l_1} \tau_{l_2} \tau_{l_3}$ . The fusion coefficients are invariant under permutations of the  $l_i$ , and have the additional symmetry property  $N_{l_1, l_2, l_3}^{(k)} = N_{l_1, k-l_2, k-l_3}^{(k)}$  (coming from  $\tau_{k-l} = \tau_k \tau_l$ ). One has,

$$\tau_{l_1} \tau_{l_2} = \sum_{l_3=0}^k N_{l_1, l_2, l_3}^{(k)} \tau_{l_3}.$$

Let  $\Delta \subset [0, 1]^3$  be the *Jeffrey-Weitsman polytope*, cut out by the inequalities

$$u_3 \leq u_1 + u_2, \quad u_1 \leq u_2 + u_3, \quad u_2 \leq u_3 + u_1, \quad u_1 + u_2 + u_3 \leq 2.$$

Suppose  $\mathcal{C}_i$ ,  $i = 1, 2, 3$  are conjugacy classes of elements  $\exp(u_i \rho)$ . As shown by Jeffrey-Weitsman [28, Proposition 3.1], the set  $\{g_1 g_2 g_3 \mid g_i \in \mathcal{C}_i\}$  contains  $e$  if and only if  $(u_1, u_2, u_3) \in \Delta$ . Similarly,

$$N_{l_1, l_2, l_3}^{(k)} = \begin{cases} 1 & \text{if } l_1 + l_2 + l_3 \text{ even, } (\frac{l_1}{k}, \frac{l_2}{k}, \frac{l_3}{k}) \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

### 3. THE TWISTED EQUIVARIANT $K$ -HOMOLOGY OF $SU(2)$

We will follow the approach to twisted  $K$ -homology via Dixmier-Douady bundles.

**3.1.  $G$ -Dixmier-Douady bundles.** Suppose  $G$  is a compact Lie group, acting on a (reasonable) topological space  $X$ . A  $G$ -Dixmier-Douady bundle over  $X$  is a  $G$ -equivariant bundle  $\mathcal{A} \rightarrow X$  of  $*$ -algebras, with typical fiber  $\mathbb{K}(\mathcal{H})$  the compact operators on a separable Hilbert space  $\mathcal{H}$ , and structure group  $\text{Aut}(\mathbb{K}(\mathcal{H})) = \text{PU}(\mathcal{H})$  the projective unitary group. Here  $\mathcal{H}$  is allowed to be finite-dimensional. A *Morita isomorphism* between two such bundles  $\mathcal{A}_1, \mathcal{A}_2 \rightarrow X$  is a  $G$ -equivariant bundle of  $\mathcal{A}_2 - \mathcal{A}_1$ -bimodules  $\mathcal{E} \rightarrow X$ , such that  $\mathcal{E}$  is locally modeled on the  $\mathbb{K}(\mathcal{H}_2) - \mathbb{K}(\mathcal{H}_1)$ -bimodule  $\mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$  of compact operators from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ . We write

$$\mathcal{A}_1 \simeq_{\mathcal{E}} \mathcal{A}_2.$$

One then also has  $\mathcal{A}_2 \simeq_{\mathcal{E}^{\text{op}}} \mathcal{A}_1$ , where the opposite bimodule  $\mathcal{E}^{\text{op}}$  is modeled on  $\mathbb{K}(\mathcal{H}_2, \mathcal{H}_1)$ . Any two Morita isomorphisms  $\mathcal{E}, \mathcal{E}'$  between  $\mathcal{A}_1, \mathcal{A}_2$  differ by a  $G$ -equivariant line bundle  $J$ , given as the bundle of bimodule homomorphisms:

$$J = \text{Hom}_{\mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}, \mathcal{E}'), \quad \mathcal{E}' = \mathcal{E} \otimes J.$$

Two equivariant Morita isomorphisms  $\mathcal{E}, \mathcal{E}'$  will be called *equivalent* if this line bundle is equivariantly trivial. By the Dixmier-Douady theorem [15] (extended to the equivariant case by Atiyah-Segal [7]), the Morita isomorphism classes of  $G$ -Dixmier-Douady bundles  $\mathcal{A} \rightarrow X$  are classified by an equivariant *Dixmier-Douady class*  $\text{DD}_G(\mathcal{A}) \in H_G^3(X, \mathbb{Z})$ . Put differently, the Dixmier-Douady class is the obstruction to an equivariant Morita trivialization  $\mathbb{C} \simeq_{\mathcal{E}} \mathcal{A}$ , i.e. an equivariant Hilbert space bundle  $\mathcal{E}$  with an isomorphism  $\mathcal{A} \cong \mathbb{K}(\mathcal{E})$ .

*Remark 3.1.* For  $G = \{e\}$  the Dixmier-Douady class is realized as a Čech cohomology class, as follows: Choose a cover  $\{U_a\}$  of  $M$  with Morita trivialization  $\mathbb{C} \simeq_{\mathcal{E}_a} \mathcal{A}|_{U_a}$ . On overlaps, the  $\mathcal{E}_a$  are related by ‘transition line bundles’,

$$J_{ab} = \text{Hom}_{\mathcal{A}}(\mathcal{E}_a, \mathcal{E}_b), \quad \mathcal{E}_b = \mathcal{E}_a \otimes J_{ab}.$$

On triple overlaps, one has a trivializing section  $\theta_{abc}$  of  $J_{ab} \otimes J_{bc} \otimes J_{ca}$ . Taking  $U_a$  sufficiently fine, the  $J_{ab}$  are all trivial, and a choice of trivialization makes  $\theta_{abc}$  into a collection of  $U(1)$ -valued functions defining a Čech cocycle. A different choice of trivialization of the  $J_{ab}$  changes the cocycle by a coboundary. The class  $\text{DD}(\mathcal{A})$  equals the cohomology class of  $\theta$ , under the isomorphism  $H^2(X, \underline{U(1)}) = H^3(X, \mathbb{Z})$ .

**3.2. The Dixmier-Douady bundle over SU(2).** We will now give a fairly explicit construction of an equivariant Dixmier-Douady bundle representing the generator of  $H_{\text{SU}(2)}^3(\text{SU}(2), \mathbb{Z}) = \mathbb{Z}$ , using the cover (1). Let  $\mathcal{H}$  be any SU(2)-Hilbert space, with the property that  $\mathcal{H}$  contains all  $T$ -weights with infinite multiplicity. (A possible choice is  $\mathcal{H} = L^2(\text{SU}(2))$  with the left regular representation.) As a consequence, there exists a  $T$ -equivariant unitary isomorphism,

$$(4) \quad \mathcal{H} \rightarrow \mathcal{H} \otimes \mathbb{C}_\rho$$

(given by a collection of isomorphisms of the  $\mu$ -weight spaces with the  $\mu - \rho$ -weight spaces). Let

$$\mathcal{E}_\pm = \text{SU}(2)_\pm \times \mathcal{H}$$

with the diagonal SU(2)-action. By (2), any SU(2)-equivariant bundle over  $\text{SU}(2)_{\text{reg}}$  is uniquely determined by its restriction to a  $T$ -equivariant bundle over  $T_{(0,\rho)}$ . Let  $J \rightarrow \text{SU}(2)_{\text{reg}}$  be the equivariant line bundle such that  $J|_{T_{(0,\rho)}} = T_{(0,\rho)} \times \mathbb{C}_\rho$ . The isomorphism (4) defines a  $T$ -equivariant isomorphism

$$\mathcal{E}_-|_{T_{(0,\rho)}} \rightarrow \mathcal{E}_+|_{T_{(0,\rho)}} \otimes J|_{T_{(0,\rho)}}$$

which extends to an SU(2)-equivariant isomorphism  $\mathcal{E}_-|_{\text{SU}(2)_{\text{reg}}} \rightarrow \mathcal{E}_+|_{\text{SU}(2)_{\text{reg}}} \otimes J$ . This then defines an isomorphism  $\mathbb{K}(\mathcal{E}_-)|_{\text{SU}(2)_{\text{reg}}} \rightarrow \mathbb{K}(\mathcal{E}_+)|_{\text{SU}(2)_{\text{reg}}}$ , which we use to glue  $\mathbb{K}(\mathcal{E}_\pm)$  to a global bundle  $\mathcal{A}$ . The bundle  $\mathcal{A}$  represents the generator of  $H_{\text{SU}(2)}^3(\text{SU}(2), \mathbb{Z}) = \mathbb{Z}$ . Since  $H_{\text{SU}(2)}^2(\text{SU}(2), \mathbb{Z}) = 0$ , any other Dixmier-Douady bundle  $\mathcal{A}'$  representing the generator is related to  $\mathcal{A}$  by a *unique* (up to equivalence) Morita isomorphism. Again, this can be made quite explicit: Let  $\mathcal{E}'_\pm$  be Morita trivializations of  $\mathcal{A}'$ , with transition line bundle  $J'$ . Then the Morita  $\mathcal{A} - \mathcal{A}'$  bimodule is obtained by gluing  $\mathbb{K}(\mathcal{E}'_+, \mathcal{E}_+)$  with  $\mathbb{K}(\mathcal{E}'_-, \mathcal{E}_-)$ , where the isomorphism over  $\text{SU}(2)_{\text{reg}}$  is defined by the choice of an equivariant isomorphism  $J' \cong J$  (the latter is unique up to homotopy).

**3.3. The equivariant Cartan 3-form on SU(2).** The equivariant Dixmier-Douady bundle  $\mathcal{A} \rightarrow \text{SU}(2)$  may be viewed as a ‘pre-quantization’ of the generator of equivariant Cartan 3-form on SU(2). To explain this viewpoint, we need some notation. For any manifold  $M$  with an action of a Lie group  $G$ , we denote by  $\xi_M \in \mathfrak{X}(M)$ ,  $\xi \in \mathfrak{g}$  the generating vector fields for the infinitesimal  $\mathfrak{g}$ -action. That is,  $\xi_M(f) = \frac{\partial}{\partial u}|_{u=0}(\exp(-u\xi))^*f$  for  $f \in C^\infty(M)$ . We let  $(\Omega_G^\bullet(M), d_G)$  denote the complex of equivariant differential forms

$$\Omega_G^k(M) = \bigoplus_{2i+j=k} (S^i \mathfrak{g}^* \otimes \Omega^j(M))^G,$$

with equivariant differential  $(d_G \gamma)(\xi) = d\gamma(\xi) - \iota(\xi_M)\gamma(\xi)$ . For  $G$  compact, its cohomology is identified with Borel’s equivariant cohomology  $H_G^k(M, \mathbb{R})$ .

Let  $\theta^L, \theta^R \in \Omega^1(\text{SU}(2), \mathfrak{su}(2))$  be the Maurer-Cartan forms on SU(2). The *Cartan 3-form*  $\eta \in \Omega^3(\text{SU}(2))$  is given in terms of the basic inner product  $\cdot$  on  $\mathfrak{su}(2)$  by

$$\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L].$$

It is d-closed, and has an equivariantly closed extension  $\eta_{\text{SU}(2)} \in \Omega_{\text{SU}(2)}^3(\text{SU}(2))$ ,

$$\eta_{\text{SU}(2)}(\xi) = \eta - \frac{1}{2}(\theta^L + \theta^R) \cdot \xi.$$

Let  $\varpi \in \Omega^2(\mathfrak{su}(2))$  be the invariant primitive of  $\exp^* \eta$  defined by the de Rham homotopy operator for the radial homotopy. The image of the (non-closed) 2-form  $d\mu - \frac{1}{2} \exp^*(\theta^L + \theta^R)$  under the homotopy operator is zero, since its pull-back to any line through the origin vanishes. Hence

$$(5) \quad \exp^* \eta_{\mathrm{SU}(2)} = d_{\mathrm{SU}(2)}(\varpi - \mu)$$

where the ‘identity function’  $\mu: \mathfrak{g} \rightarrow \mathfrak{g}$  is viewed as an element of  $\mathfrak{su}(2)^* \otimes \Omega^0(\mathfrak{su}(2))$ .

**Lemma 3.2.** *For any  $G$ -manifold with a closed equivariant 3-form  $\gamma \in \Omega_G^3(M)$ , all  $G$ -orbits  $S \subset M$  acquire unique invariant 2-forms  $\omega_S \in \Omega^2(S)^G$  such that  $d_G \omega_S = i_S^* \gamma$ .*

The straightforward proof is left to the reader. As special cases, we obtain 2-forms  $\omega_{\mathcal{C}}$  on the conjugacy classes  $\mathcal{C} \subset \mathrm{SU}(2)$  and  $\omega_{\mathcal{O}}$  on the adjoint orbits  $\mathcal{O} \subset \mathfrak{su}(2)$  such that

$$d_{\mathrm{SU}(2)} \omega_{\mathcal{C}} = -\iota_{\mathcal{C}}^* \eta_{\mathrm{SU}(2)}, \quad d_{\mathrm{SU}(2)} \omega_{\mathcal{O}} = \iota_{\mathcal{O}}^*(d\mu).$$

Under the identification of  $\mathfrak{su}(2)$  with its dual,  $\omega_{\mathcal{O}}$  is just the usual symplectic form on co-adjoint orbits. Suppose  $\mathcal{C} = \exp(\mathcal{O})$ . Then (5) and the uniqueness part of the Lemma imply

$$(6) \quad i_{\mathcal{O}}^* \varpi = \omega_{\mathcal{O}} - (\exp|_{\mathcal{O}})^* \omega_{\mathcal{C}}.$$

Let  $V \subset \mathfrak{su}(2)$  be the open ball of radius  $\frac{1}{\sqrt{2}}$ . We have diffeomorphisms

$$\exp_{\pm}: V \cong \mathrm{SU}(2)_{\pm}$$

where  $\exp_+$  is the restriction of the exponential map, and  $\exp_- = l_c \circ \exp_+$  is its left translate by the central element  $c$ . The inverse maps will be denoted

$$\log_{\pm}: \mathrm{SU}(2)_{\pm} \rightarrow V \subset \mathfrak{su}(2).$$

Let  $\varpi_{\pm} = \log_{\pm}^* \varpi \in \mathrm{SU}(2)_{\pm}$ . Then  $d\varpi_{\pm} = \eta$  over  $\mathrm{SU}(2)_{\pm}$ . Furthermore, by Equation (5) we have, over  $\mathrm{SU}(2)_{\pm}$ ,

$$(7) \quad d_{\mathrm{SU}(2)}(\varpi_{\pm} - \log_{\pm}) = \eta_{\mathrm{SU}(2)}.$$

Over  $\mathrm{SU}(2)_{\mathrm{reg}}$ , both  $\varpi_{\pm}$  are primitives of  $\eta$ , hence their difference is closed. To determine this closed 2-form, recall (cf. Equation (2)) that  $\mathrm{SU}(2)_{\mathrm{reg}} \cong T_{(0,\rho)} \times \mathrm{SU}(2)/T$ . Let

$$\Psi: \mathrm{SU}(2)_{\mathrm{reg}} \rightarrow \mathrm{SU}(2)/T$$

be the projection to the second factor, and identify  $\mathrm{SU}(2)/T$  with the (co)-adjoint orbit  $\mathcal{O} = \mathrm{SU}(2) \cdot \rho$ .

**Lemma 3.3.** *One has  $\varpi_- - \varpi_+ = \Psi^* \omega_{\mathcal{O}}$  over  $\mathrm{SU}(2)_{\mathrm{reg}}$ , where  $\mathcal{O}$  is the adjoint orbit of the element  $\rho$ .*

*Proof.* By (7) we have

$$d_{\mathrm{SU}(2)}(\varpi_- - \varpi_+ - (\log_- - \log_+)) = 0$$

over  $\mathrm{SU}(2)_{\mathrm{reg}}$ . Thus,  $\log_+ - \log_-$  serves as a moment map for the closed invariant 2-form  $\varpi_- - \varpi_+$ . We claim

$$\log_+ - \log_- = \iota_{\mathcal{O}} \circ \Psi.$$

Since both sides are  $\mathrm{SU}(2)$ -equivariant, it suffices to compare the restrictions to  $T_{(0,\rho)} \subset \mathrm{SU}(2)_{\mathrm{reg}}$ . Indeed,  $\log_+(\exp(u\rho)) = u\rho$  and  $\log_- \exp(u\rho) = \log(\exp(u -$

$1)\rho) = (u - 1)\rho$ , so the difference is  $(\log_+ - \log_-)(\exp(u\rho)) = \rho$  as needed. This gives

$$0 = d_{SU(2)}(\varpi_- - \varpi_+ + \iota_{\mathcal{O}} \circ \Psi) = d_{SU(2)}(\varpi_- - \varpi_+ - \Psi^* \omega_{\mathcal{O}})$$

In particular,  $\varpi_- - \varpi_+ - \Psi^* \omega_{\mathcal{O}}$  is annihilated by all contractions with generating vector fields for the conjugation action. It is hence enough to show that its pull-back to  $T_{(0,\rho)}$  is zero. Indeed, by applying the homotopy operator to  $\exp_T^* \iota_T^* \eta_{SU(2)} = 0$ , we see that  $\iota_t^* \varpi = 0$ , which implies that  $\varpi_{\pm}$  pull back to 0 on  $T$ .  $\square$

The 2-form  $\omega_{\mathcal{O}}$  is the curvature form  $\text{curv}(\nabla)$  of the line bundle  $SU(2) \times_T \mathbb{C}_{\rho}$ , for the unique invariant connection  $\nabla$  on this bundle. Let  $J = \Psi^*(SU(2) \times_T \mathbb{C}_{\rho})$  carry the pull-back connection  $\nabla_J$ . The identities

$$\varpi_- - \varpi_+ = \text{curv}(\nabla_J), \quad d\varpi_{\pm} = \eta$$

say that  $(\nabla_J, \varpi_{\pm})$  is a ‘gerbe connection’ in the sense of Chatterjee-Hitchin [13, 25], with  $\eta$  as its 3-curvature. Similarly,  $(\nabla_J, \varpi_{\pm} - \log_{\pm})$  is an equivariant gerbe connection, with equivariant 3-curvature  $\eta_{SU(2)}$ .

We conclude this Section with an easy proof of the fact that  $\eta$  integrates to 1. Observe that  $\partial V = \overline{V} \setminus V$  is the (co-)adjoint orbit  $\mathcal{O}$  of the element  $\rho$ . It has symplectic volume  $\int_{\mathcal{O}} \omega_{\mathcal{O}} = 1$  by the well-known formula for volume of coadjoint orbits [12, Corollary 7.27]. Since  $\mathcal{C} := \exp \mathcal{O} = \{c\}$ , we have  $\omega_{\mathcal{C}} = 0$ . Hence Equation (6) together with Stokes’ theorem give

$$\int_{SU(2)} \eta = \int_V d\varpi = \int_{\mathcal{O}} \iota_{\mathcal{O}}^* \varpi = \int_{\mathcal{O}} \omega_{\mathcal{O}} = 1.$$

**3.4. Twisted  $K$ -homology.** Let  $G$  be a compact Lie group acting on a compact  $G$ -space  $X$ . Given a  $G$ -Dixmier-Douady bundle  $\mathcal{A} \rightarrow X$ , one defines (following J. Rosenberg [37]) the twisted  $K$ -homology group

$$K_0^G(X, \mathcal{A}) = K_0^G(\Gamma(X, \mathcal{A})),$$

where the right hand side denotes the  $K$ -homology group of the  $G - C^*$ -algebra of sections of  $\mathcal{A}$ . (For  $K$ -homology of  $C^*$ -algebras, see [24, 29].) The twisted  $K$ -homology is a covariant functor: If  $\Phi: X_1 \rightarrow X_2$  is an equivariant map of compact  $G$ -spaces, together with an equivariant Morita isomorphism  $\mathcal{A}_1 \simeq_{\mathcal{E}} \Phi^* \mathcal{A}_2$ , one obtains a push-forward map

$$\Phi_*: K_0^G(X_1, \mathcal{A}_1) \rightarrow K_0^G(X_2, \mathcal{A}_2).$$

It is possible to work out many examples of twisted equivariant  $K$ -homology groups simply from its formal properties such as excision, Poincaré duality and so on. For  $\mathcal{A} = \mathbb{C}$  one obtains the untwisted  $K$ -homology groups. One has a ring isomorphism

$$K_0^G(\text{pt}) = R(G),$$

where the ring structure on the left hand side is realized as push-forward under  $\text{pt} \times \text{pt} \rightarrow \text{pt}$ . The following is the simplest non-trivial case of the Freed-Hopkins-Teleman theorem [21]. This special case may be proved by an elementary Mayer-Vietoris argument, see Freed [20].

**Theorem 3.4.** *Let  $SU(2)$  act on itself by conjugation, and let  $\mathcal{A} \rightarrow SU(2)$  be the basic Dixmier-Douady bundle. For all levels  $k = 0, 1, 2, \dots$ , the  $R(SU(2))$ -module homomorphism*

$$R(SU(2)) \cong K_0^{SU(2)}(\text{pt}) \rightarrow K_0^{SU(2)}(SU(2), \mathcal{A}^{k+2})$$

given as push-forward under the inclusion of the group unit  $\text{pt} \rightarrow \text{SU}(2)$  is onto, with kernel the level  $k$  fusion ideal  $I_k(\text{SU}(2))$ . It hence defines a ring isomorphism,

$$R_k(\text{SU}(2)) \cong K_0^{\text{SU}(2)}(\text{SU}(2), \mathcal{A}^{k+2}).$$

**3.5. The  $K$ -homology fundamental class.** Recall that for  $n$  even, the complex Clifford algebra  $\mathbb{C}l(n) = \mathbb{C}l(\mathbb{R}^n)$  admits a unique (up to isomorphism) irreducible  $*$ -representation. Concretely, the identification  $\mathbb{R}^n \cong \mathbb{C}^{n/2}$  gives a Clifford action on the standard spinor module  $\mathcal{S} = \wedge \mathbb{C}^{n/2}$ . This realizes the Clifford algebra as a matrix algebra,  $\mathbb{C}l(n) = \text{End}(\mathcal{S})$ . Given  $A \in \text{SO}(n)$  there exists a unitary transformation  $U \in \text{U}(\mathcal{S})$ , unique up to a scalar, such that  $A(v).U(z) = U(v.z)$  for  $v \in \mathbb{R}^n$ ,  $z \in \mathcal{S}$ . The set of such *implementers*  $U$  forms a closed subgroup of  $\text{U}(\mathcal{S})$ , denoted  $\text{Spin}_c(n)$ , and the map taking  $U$  to  $A$  makes this group into a central extension

$$1 \rightarrow \text{U}(1) \rightarrow \text{Spin}_c(n) \rightarrow \text{SO}(n) \rightarrow 1.$$

If  $M$  is an oriented Riemannian  $G$ -manifold of even dimension  $n$ , then its Clifford algebra bundle  $\mathbb{C}l(TM)$  is a  $G$ -equivariant bundle of complex matrix algebras. It is thus a  $G$ -Dixmier-Douady bundle. Its Dixmier-Douady class is the third integral equivariant <sup>1</sup> Stiefel-Whitney class,  $W_G^3(M) \in H_G^3(M, \mathbb{Z})$ . As pointed out by Connes [14] and Plymen [35], an equivariant  $\text{Spin}_c$ -structure on  $M$  is exactly the same thing as an equivariant Morita trivialization of  $\mathbb{C}l(TM)$ . Indeed, given an equivariant lift  $P_{\text{Spin}_c}(M) \rightarrow P_{\text{SO}}(M)$  of the  $\text{SO}(n)$ -frame bundle to the group  $\text{Spin}_c(n)$ , the Morita trivialization is defined by the bundle of spinors  $\mathcal{S} = P_{\text{Spin}_c}(M) \times_{\text{Spin}_c(n)} \mathcal{S}$ . Conversely, given an equivariant Morita trivialization  $\mathbb{C}l(TM) \simeq_{\mathcal{S}} \mathbb{C}$ , one obtains a lift of the structure group: The fiber of the bundle  $P_{\text{Spin}_c}(M)$  at  $m \in M$  is the set of pairs  $(A, U)$ , where  $A: T_m M \rightarrow \mathbb{R}^n$  is an oriented orthonormal frame, and  $U: \mathcal{S}_m \rightarrow \mathcal{S}$  is a unitary isomorphism intertwining the Clifford actions of  $v \in T_m M$  and  $A(v) \in \mathbb{R}^n$ .

The Clifford bundle  $\mathbb{C}l(TM)$  is naturally a  $\mathbb{C}l(TM) - \mathbb{C}l(TM)$  bimodule. Using the canonical anti-automorphism of  $\mathbb{C}l(TM)$ , it may also be viewed as a module over  $\mathbb{C}l(TM) \otimes \mathbb{C}l(TM)$ , defining a Morita trivialization of the latter. Given any  $\text{Spin}_c$ -structure  $\mathcal{S}$ , one obtains a Hermitian line bundle

$$\mathcal{L} := \mathcal{L}(\mathcal{S}) = \text{Hom}_{\mathbb{C}l(TM) \otimes \mathbb{C}l(TM)}(\mathbb{C}l(TM), \mathcal{S} \otimes \mathcal{S})$$

called the  $\text{Spin}_c$ -line bundle. Twisting  $\mathcal{S}$  by a line bundle  $L$  changes the  $\text{Spin}_c$ -line bundle as follows,

$$\mathcal{L}(\mathcal{S} \otimes L) = \mathcal{L}(\mathcal{S}) \otimes L^2.$$

For any equivariant  $\text{Spin}_c$ -structure on an even-dimensional manifold, the class of the  $\text{Spin}_c$ -Dirac operator defines a *fundamental class* in equivariant  $K$ -homology. In the absence of a  $\text{Spin}_c$ -structure, there is still a fundamental class, but as an element

$$[M] \in K_0^G(M, \mathbb{C}l(TM))$$

---

<sup>1</sup>We remark that for  $G$  compact and simply connected, the vanishing of  $W_G^3(M)$  is equivalent to the vanishing of the non-equivariant Stiefel-Whitney class  $W^3(M)$ , since the map  $H_G^3(M, \mathbb{Z}) \rightarrow H^3(M, \mathbb{Z})$  is injective (cf. [30]).



in twisted  $K$ -homology.<sup>2</sup> For an explicit construction of  $[M]$ , see Kasparov [29]. Below, we will construct elements of  $R_k(\mathrm{SU}(2)) = K_0^{\mathrm{SU}(2)}(\mathrm{SU}(2), \mathcal{A}^{k+2})$  as push-forwards of  $[M]$  under  $\mathrm{SU}(2)$ -equivariant maps  $\Phi: M \rightarrow \mathrm{SU}(2)$ . In order to define such a push-forward, we need an equivariant Morita isomorphism

$$\mathbb{C}l(TM) \simeq_{\mathcal{E}} \Phi^* \mathcal{A}^{k+2}.$$

We will explain how such a ‘twisted  $\mathrm{Spin}_c$ -structure’ arises for pre-quantized q-Hamiltonian  $\mathrm{SU}(2)$ -spaces. The counterpart to the  $\mathrm{Spin}_c$ -line bundle is the Morita isomorphism  $\Phi^* \mathcal{A}^{2k+4} \simeq_{\mathcal{K}} \mathbb{C}$  given by

$$\mathcal{K} = \mathrm{Hom}_{\mathbb{C}l(TM) \otimes \mathbb{C}l(TM)}(\mathbb{C}l(TM), (\mathcal{E} \otimes \mathcal{E})^{\mathrm{op}}).$$

#### 4. Q-HAMILTONIAN SU(2)-SPACES

**4.1. Basic definitions.** Let  $G$  be a compact Lie group, with Lie algebra  $\mathfrak{g}$ . Given an invariant inner product  $B$  on its Lie algebra, define the equivariant Cartan 3-form

$$\eta_G^{(B)}(\xi) = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) - \frac{1}{2} B(\theta^L + \theta^R, \xi).$$

A *q-Hamiltonian  $G$ -space* (relative to the inner product  $B$ ) is a triple  $(M, \omega, \Phi)$  where  $M$  is a  $G$ -manifold,  $\omega$  is an invariant 2-form, and  $\Phi: M \rightarrow G$  an equivariant smooth map, called the *moment map*, such that

- (i)  $d_G \omega = -\Phi^* \eta_G^{(B)}$ ,
- (ii)  $\ker \omega \cap \ker(d\Phi) = 0$  everywhere.

*Remark 4.1.* If  $G = T$  is a torus, this is just the usual definition of a symplectic  $T$ -space with torus-valued moment map. Indeed, Condition (i) in this case says  $d\omega = 0$  and  $\omega_m(\xi_M(m), v) = -B(\theta_T(d_m \Phi(v)), \xi)$  for all  $\xi \in \mathfrak{g}$ ,  $v \in T_m M$ . Hence it implies  $\ker(\omega) \subset \ker(d\Phi)$ , whence (ii) simplifies to  $\ker(\omega) = \{0\}$ . For general  $G$ , a similar argument shows that  $\ker(\omega_m)$  is spanned by all  $\xi_M(m)$  such that  $\mathrm{Ad}_{\Phi(m)} \xi + \xi = 0$ .

Basic examples of q-Hamiltonian  $G$ -spaces are the conjugacy classes  $\mathcal{C} \subset G$ , with moment map the embedding. The *double*  $D(G) = G \times G$ , with  $G$  acting by conjugation and with moment  $\Phi(a, b) = aba^{-1}b^{-1}$ , is another example. The 2-form is,

$$\omega = \frac{1}{2} a^* \theta^L \cdot b^* \theta^R + \frac{1}{2} a^* \theta^R \cdot b^* \theta^L + \frac{1}{2} (ab)^* \theta^L \cdot (a^{-1}b^{-1})^* \theta^R,$$

where, for example,  $a^{-1}b^{-1}$  denotes the map  $(a, b) \mapsto a^{-1}b^{-1}$ . If  $G'$  is the quotient of  $G$  by a finite subgroup of  $Z(G)$ , then the moment map, action and 2-form on  $D(G)$  descends to  $D(G')$ , so that  $D(G')$  is again a q-Hamiltonian  $G$ -space.

Given two q-Hamiltonian  $G$ -spaces  $(M_i, \omega_i, \Phi_i)$ ,  $i = 1, 2$ , their product  $M_1 \times M_2$  with the diagonal  $G$ -action, moment map  $\Phi_1 \Phi_2$ , and 2-form  $\omega_1 + \omega_2 + \frac{1}{2} B(\Phi_1^* \theta^L, \Phi_2^* \theta^R)$  is again a q-Hamiltonian  $G$ -space. This is called the *fusion product* of  $M_1, M_2$ . The *symplectic quotient* of a q-Hamiltonian  $G$ -space is  $M//G = \Phi^{-1}(e)/G$ . Similar to the Hamiltonian theory,  $e$  is a regular value of  $\Phi$  if and only if  $G$  acts locally freely on  $\Phi^{-1}(e)$ , and in this case  $M//G$  is a symplectic orbifold. (If  $e$  is a singular value, then  $M//G$  is a singular symplectic space as defined in [39].) More generally, given a conjugacy class  $\mathcal{C}$  one can define a symplectic quotient

$$M//_{\mathcal{C}} G = (M \times \mathcal{C})//G.$$

<sup>2</sup>More precisely, one has to view  $\mathbb{C}l(TM)$  as a  $\mathbb{Z}_2$ -graded Dixmier-Douady bundle, and work with the twisted  $K$ -homology for such  $\mathbb{Z}_2$ -graded bundles.

It was shown in [1] that moduli spaces of flat  $G$ -bundles over compact oriented surfaces  $\Sigma_h^r$  of genus  $h$  with  $r$  boundary circles, with boundary holonomies in prescribed conjugacy classes  $\mathcal{C}_j$ , are symplectic quotients

$$M(\Sigma_h^r, \mathcal{C}_1, \dots, \mathcal{C}_r) = \underbrace{(D(G) \times \dots \times D(G))}_{h \text{ times}} \times \mathcal{C}_1 \times \dots \times \mathcal{C}_r // G.$$

We now specialize to  $q$ -Hamiltonian  $\mathrm{SU}(2)$ -spaces  $(M, \omega, \Phi)$ , with  $B$  the basic inner product. Put  $M_\pm = \Phi^{-1}(\mathrm{SU}(2)_\pm)$ , and let

$$\begin{aligned} \omega_{0,\pm} &= \omega + \Phi^* \varpi_\pm, \\ \Phi_{0,\pm} &= \log_\pm \circ \Phi. \end{aligned}$$

Then

$$d_{\mathrm{SU}(2)}(\omega_{0,\pm} - \Phi_{0,\pm}) = d_{\mathrm{SU}(2)}(\omega + \Phi^*(\varpi_\pm - \log_\pm)) = 0.$$

That is,  $\omega_{0,\pm}$  is closed, with  $\Phi_{0,\pm}$  as a moment map. Using condition (ii) above one can show [1] that  $\omega_{0,\pm}$  are non-degenerate, i.e. *symplectic*. Thus,  $(M_\pm, \omega_{0,\pm}, \Phi_{0,\pm})$  are ordinary (symplectic) Hamiltonian  $\mathrm{SU}(2)$ -spaces. In particular,  $M_\pm$  are even-dimensional, with a natural orientation. If  $M$  is compact and connected, then the spaces  $M_\pm$  are connected. (This follows from the convexity properties and the fiber connectivity of group-valued moment maps [1].)

Conversely,  $(M, \omega, \Phi)$  is determined by the pair of Hamiltonian  $\mathrm{SU}(2)$ -spaces  $(M_\pm, \omega_{0,\pm}, \Phi_{0,\pm})$ . This correspondence reduces many properties of  $q$ -Hamiltonian spaces to standard facts about ordinary Hamiltonian spaces. It is also used to construct  $q$ -Hamiltonian spaces, as in the following example.

**4.2. Example: The 4-sphere.** The following construction of a  $q$ -Hamiltonian structure of  $S^4$  is taken from [6]. An independent construction due to Hurtubise-Jeffrey [27] was later generalized by Hurtubise-Jeffrey-Sjamaar [26] to define the structure of a  $q$ -Hamiltonian  $\mathrm{SU}(n)$ -space on  $S^{2n}$ , for any  $n$ .

Let  $\mathbb{C}^2$  carry the standard  $\mathrm{SU}(2)$ -action and the standard symplectic structure  $\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ . The moment map for the  $\mathrm{SU}(2)$ -action can be written, for  $z \neq 0$ , as

$$\Phi_0(z) = -i\pi^2 \|z\|^2 P(z) + i\pi^2 \|z\|^2 (I - P(z)),$$

where  $P(z)$  is the projection operator,

$$P(z) = \|z\|^{-2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^\dagger = \frac{1}{\|z\|^2} \begin{pmatrix} |z_1|^2 & z_1 \bar{z}_2 \\ \bar{z}_1 z_2 & |z_2|^2 \end{pmatrix}.$$

Hence,

$$\exp(\Phi_0(z)) = e^{-i\pi^2 \|z\|^2} P(z) + e^{i\pi^2 \|z\|^2} (I - P(z)).$$

Let  $V \subset \mathfrak{su}(2)$  be the open ball of radius  $\frac{1}{\sqrt{2}}$  (cf. Section 3.3). We have  $\|\Phi_0(z)\| = \frac{1}{\sqrt{2}} \pi \|z\|^2$ , so that

$$S_\pm^4 := \Phi_0^{-1}(V) = \{z \in \mathbb{C}^2 \mid \pi \|z\|^2 < 1\}.$$

Define a diffeomorphism  $F$  of the annulus  $0 < \pi \|z\|^2 < 1$  by

$$F(z_1, z_2) = (-\bar{z}_2, \bar{z}_1) \sqrt{\frac{1}{\pi \|z\|^2} - 1}.$$

Then  $F$  is equivariant, with  $\pi \|F(z)\|^2 = 1 - \pi \|z\|^2$ . Gluing the charts  $S_\pm^4$  under  $F$  one obtains a 4-sphere  $S^4$  with an action of  $\mathrm{SU}(2)$ .

Put  $\Phi_+ = \exp \Phi_0$  and  $\Phi_- = l_c \circ \exp \Phi_0 = -\exp \Phi_0$ . The diffeomorphism  $F$  satisfies  $P(F(z)) = I - P(z)$ , and therefore,

$$\Phi_+(F(z)) = \exp(\Phi_0(F(z))) = -\exp(\Phi_0(z)) = \Phi_-(z).$$

Hence  $\Phi_{\pm}$  glue to a global equivariant map  $\Phi: S^4 \rightarrow \text{SU}(2)$ . Similarly, the 2-forms  $\omega_{\pm} = \omega_0 + \Phi_0^* \varpi$  glue<sup>3</sup> to a global invariant 2-form  $\omega \in \Omega^2(S^4)$ , defining a q-Hamiltonian SU(2)-space  $(S^4, \omega, \Phi)$ .

*Remark 4.2.* The space  $S^4$  carries an involution  $I: S^4 \rightarrow S^4$ , given in charts by the complex conjugation. It has the equivariance property  $I(g.x) = I(g).I(x)$  relative to the involution of SU(2) given by complex conjugation of matrices,  $I(A) = \overline{A}$ . The involution satisfies,  $I^*\omega = -\omega$  and  $I^*\Phi = \overline{\Phi}$ . The fixed point set of the involution is a 2-sphere  $S^2 \subset S^4$ . The theory of anti-involutions of  $q$ -Hamiltonian  $G$ -spaces was developed in recent work of Schaffhauser [38], who established an analogue of the convexity results of Duistermaat [16] and O'Shea-Sjamaar [33] in this context.

*Remark 4.3.* It is well-known that the complement of the zero section in  $T^*(S^2)$  is SU(2)-equivariantly symplectomorphic to the complement of the origin in  $\mathbb{C}^2$ . One may thus modify the construction above, and obtain examples where the fiber over  $e$  or over  $c$  (or both) is a 2-sphere rather than a point. The four examples obtained in this way are the complete list of 4-dimensional q-Hamiltonian SU(2)-spaces with surjective moment map.

## 5. CROSS-SECTIONS

Let  $(M, \omega, \Phi)$  be a q-Hamiltonian SU(2)-space. By the q-Hamiltonian cross-section theorem [1], the pre-image

$$(8) \quad Y = \Phi^{-1}(T_{(0,\rho)})$$

is a q-Hamiltonian  $T$ -space  $(Y, \omega_Y, \Phi|_Y)$ , with 2-form  $\omega_Y = i_Y^* \omega$ . In particular,  $\omega_Y$  is symplectic. Letting  $\Phi_Y: Y \rightarrow (0, \rho) \subset \mathfrak{t}$  with  $\exp \Phi_Y = \Phi|_Y$ , it is immediate that  $(Y, \omega_Y, \Phi_Y)$  is an ordinary Hamiltonian  $T$ -space. We have,

$$M_{\text{reg}} = M_+ \cap M_- = \text{SU}(2) \times_T Y$$

and

$$TM|_Y = TY \oplus \mathfrak{t}^{\perp},$$

where the second summand is embedded by the generating vector fields. This splitting is  $\omega$ -orthogonal, and the 2-form on  $Y \times \mathfrak{t}^{\perp}$  is given at  $y \in Y$ , with  $g = \Phi(y) \in T_{(0,\rho)}$ , by  $(\xi_1, \xi_2) \mapsto \frac{1}{2}((\text{Ad}_g - \text{Ad}_{g^{-1}})\xi_1, \xi_2)$ . Note that since the pull-back of  $\varpi_{\pm}$  to  $T_{(0,\rho)}$  is zero, the 2-forms  $\omega_{0,\pm}$  both pull back to  $\omega_Y$ . Similarly

$$\Phi_{0,+}|_Y = \Phi_Y = \Phi_{0,-}|_Y + \rho.$$

That is,  $(Y, \omega_Y, \Phi_Y)$  may also be viewed as symplectic cross-section of  $M_{\pm}$ . (To be precise, in the case of  $M_-$ , it is the *opposite* cross-section, given as the pre-image of  $(-\infty, 0) \subset \mathfrak{t}$  under  $\Phi_{0,-}$ .) The 2-forms on the bundles  $Y \times \mathfrak{t}^{\perp}$  induced by  $\omega_{0,\pm}$  are,

$$(\xi_1, \xi_2) \mapsto \text{ad}_{\mu_{\pm}} \xi_1 \cdot \xi_2,$$

where  $\mu_+ = \Phi_{0,+}(y)$  and  $\mu_- = \Phi_{0,-}(y)$ .

---

<sup>3</sup>To check that these 2-forms agree on the overlap  $S_{\text{reg}}^4 = S_+^4 \cap S_-^4$ , it suffices to consider their pull-back to symplectic cross-sections as in Section 5.

The space  $Y$  is only a ‘partial’ cross-section for  $M$ , since it leaves out the subsets  $\Phi^{-1}(e)$ ,  $\Phi^{-1}(c)$ . On the other hand, the ‘full’ cross-section  $\tilde{Y} = \Phi^{-1}(T_{[0,\rho]})$  is usually not a manifold, let alone symplectic. However, following Hurtubise-Jeffrey-Sjamaar [26] one can ‘implode’  $\tilde{Y}$  to obtain a symplectic  $T$ -space  $X$ , which is a symplectic orbifold under regularity conditions. As a topological space, the *imploded cross-section* is a quotient space

$$X = \Phi^{-1}(T_{[0,\rho]}) / \sim,$$

where the equivalence relation divides out the  $\mathrm{SU}(2)$ -action on both  $\Phi^{-1}(e)$  and on  $\Phi^{-1}(c)$ . We have a decomposition of  $X$  into three symplectic spaces,

$$(9) \quad X = (M // \mathrm{SU}(2)) \cup Y \cup (M //_c \mathrm{SU}(2))$$

The action of  $T \subset \mathrm{SU}(2)$  on  $\Phi^{-1}(T_{[0,\rho]})$  descends to an action on  $X$ , and the map  $\Phi^{-1}(T_{[0,\rho]}) \rightarrow [0, \rho] \subset \mathfrak{t}$  descends to a  $T$ -equivariant map

$$\Phi_X : X \rightarrow \mathfrak{t}.$$

Let

$$X_+ = (M // \mathrm{SU}(2)) \cup Y, \quad X_- = Y \cup (M //_c \mathrm{SU}(2)),$$

so that  $X_{\pm}$  are the imploded cross-sections of  $M_{\pm}$ . View  $M_{\pm}$  as Hamiltonian  $\mathrm{SU}(2)$ -spaces with 2-forms  $\omega_{0,\pm}$ , and let  $\mathbb{C}^2$  carry the standard structure as a Hamiltonian  $\mathrm{SU}(2)$ -space.

**Proposition 5.1.** *Suppose  $\mathrm{SU}(2)$  acts locally freely (resp. freely) on  $\Phi^{-1}(e)$ ,  $\Phi^{-1}(c)$ . Then the imploded cross-section  $X$  admits a unique structure of a symplectic orbifold (resp. symplectic manifold), such that the open subsets  $X_{\pm}$  are symplectic quotients,*

$$X_{\pm} = (M_{\pm} \times \mathbb{C}^2) // \mathrm{SU}(2).$$

Furthermore,

- (a) *The restriction of  $\Phi_X$  to  $X_{\pm}$  is smooth, and is a moment map for the action of  $T \cong \mathrm{U}(1)$ .*
- (b) *The Hamiltonian  $T$ -space  $(Y, \omega_Y, \Phi_Y)$  is embedded as an open symplectic submanifold of  $X$ .*
- (c)  *$M // \mathrm{SU}(2)$  is a symplectic suborbifold (resp. submanifold), with normal bundle  $\Phi^{-1}(e) \times_{\mathrm{SU}(2)} \mathbb{C}^2$ . The  $\mathrm{U}(1)$  action on the normal bundle is with weights  $(-1, -1)$ .*
- (d)  *$M //_c \mathrm{SU}(2)$  is a symplectic suborbifold (resp. submanifold), with normal bundle  $\Phi^{-1}(c) \times_{\mathrm{SU}(2)} \mathbb{C}^2$ . The  $\mathrm{U}(1)$ -action on the normal bundle is with weights  $(1, 1)$ .*

Thus,  $X$  is obtained by gluing the Hamiltonian imploded cross-sections for  $(M_{\pm}, \omega_{0,\pm}, \Phi_{0,\pm})$ . For the case  $G = \mathrm{SU}(2)$ , the imploded cross-sections construction was introduced by Eugene Lerman as an  $\mathrm{SU}(2)$ -counterpart of symplectic cutting. Its basis properties for Hamiltonian  $\mathrm{SU}(2)$ -spaces are described in [31, Appendix], and directly imply the properties for q-Hamiltonian  $\mathrm{SU}(2)$ -spaces.

*Remark 5.2.* More intrinsically, the imploded cross-section can directly be constructed as a q-Hamiltonian symplectic quotient  $X = (M \times S^4) // \mathrm{SU}(2)$ . This is the approach taken in [27, 26]. However, in this paper we will have more use for the construction in terms of ordinary Hamiltonian quotients.

6. THE CANONICAL ‘TWISTED  $\text{Spin}_c$ -STRUCTURE’

Choose invariant almost complex structures on  $M_\pm$ , which are compatible with  $\omega_{0,\pm}$  in the sense that each tangent space is isomorphic to  $\mathbb{C}^{n/2}$  with the standard complex structure and standard symplectic form. The almost complex structure defines spinor modules

$$\mathcal{S}_{0,\pm} = \wedge_{\mathbb{C}} TM_\pm \rightarrow M_\pm$$

for the Clifford bundles  $\mathbb{C}l(TM)|_{M_\pm}$ , where the notation  $\wedge_{\mathbb{C}}$  denotes the complex exterior powers of  $TM_\pm$  relative to the given complex structure. On the overlap  $M_+ \cap M_- = M_{\text{reg}}$ , the two spinor bundles differ by  $\text{Hom}_{\mathbb{C}l(TM)}(\mathcal{S}_{0,+}, \mathcal{S}_{0,-})$ .

**Proposition 6.1.** *The line bundle  $\text{Hom}_{\mathbb{C}l(TM)}(\mathcal{S}_{0,+}, \mathcal{S}_{0,-})$  is equivariantly isomorphic to the pull-back  $\Phi^*(J^{\otimes 2})$ .*

*Proof.* An SU(2)-invariant almost complex structure on  $M_{\text{reg}} = \text{SU}(2) \times_T Y$  is equivalent to a  $T$ -invariant complex structure on the bundle  $TM|_Y = TY \oplus \mathfrak{t}^\perp$ . This bundle carries two symplectic structures, defined by the 2-forms  $\omega_{0,\pm}$  on  $M_\pm$ . Pick a  $T$ -invariant compatible structure on the bundle  $TY$ . Its sum with the complex structure on  $\mathfrak{t}^\perp$ , coming from the identification  $\mathfrak{t}^\perp \cong \mathbb{C}_\alpha$ , is compatible with  $\omega_{0,+}$ . Similarly its sum with the complex structure on  $\mathfrak{t}^\perp$ , coming from the identification  $\mathfrak{t}^\perp \cong \mathbb{C}_{-\alpha}$ , is compatible with  $\omega_{0,-}$ . The corresponding spinor bundles  $\tilde{\mathcal{S}}_{0,\pm}|_Y \rightarrow Y$  are related by a twist by a  $T$ -equivariant line bundle, corresponding to the change of the complex structure on  $\mathfrak{t}^\perp$  to its opposite. Clearly, this is the line bundle  $Y \times \mathbb{C}_\alpha = Y \times (\mathbb{C}_\rho)^2$ :

$$\tilde{\mathcal{S}}_{0,-}|_Y = \tilde{\mathcal{S}}_{0,+}|_Y \otimes (Y \times (\mathbb{C}_\rho)^2).$$

Extending to  $M_{\text{reg}}$ , and using the definition of  $J \rightarrow \text{SU}(2)_{\text{reg}}$  we obtain

$$\tilde{\mathcal{S}}_{0,-} = \tilde{\mathcal{S}}_{0,+} \otimes \Phi^* J^2.$$

But  $\tilde{\mathcal{S}}_{0,\pm}$  are equivariantly isotopic to  $\mathcal{S}_{0,\pm}$ , since any two choices of equivariant compatible almost complex structures are isotopic. Hence we also have  $\mathcal{S}_{0,-} \cong \mathcal{S}_{0,+} \otimes \Phi^* J^2$ , or equivalently  $\text{Hom}_{\mathbb{C}l(TM)}(\mathcal{S}_{0,+}, \mathcal{S}_{0,-}) \cong \Phi^* J^2$ .  $\square$

Equivalently, we can express this result as follows:

**Proposition 6.2.** *For any  $q$ -Hamiltonian SU(2)-space  $(M, \omega, \Phi)$ , there is a distinguished (up to equivalence) SU(2)-equivariant Morita isomorphism*

$$(10) \quad \Phi^* \mathcal{A}^2 \simeq_{\mathcal{S}} \mathbb{C}l(TM),$$

*Proof.* Let  $\mathcal{F}_\pm \rightarrow \text{SU}(2)_\pm$  define Morita trivializations  $\mathbb{C} \simeq_{\mathcal{F}_\pm} \mathcal{A}^2$ . Fix isomorphisms  $\mathcal{F}_- \cong \mathcal{F}_+ \otimes J^2$  and  $\mathcal{S}_{0,-} \cong \mathcal{S}_{0,+} \otimes \Phi^* J^2$  on intersections. The desired Morita  $\mathbb{C}l(TM) - \Phi^* \mathcal{A}^2$  bimodule  $\mathcal{S}$  is then obtained by gluing the bundles  $\mathcal{S}_\pm = \text{Hom}_{\mathbb{C}}(\Phi^* \mathcal{F}_\pm, \mathcal{S}_{0,\pm})$ , using that

$$\text{Hom}_{\mathbb{C}}(\Phi^* \mathcal{F}_-, \mathcal{S}_{0,-}) \cong \text{Hom}_{\mathbb{C}}(\Phi^*(\mathcal{F}_+ \otimes J^2), \mathcal{S}_{0,+} \otimes \Phi^* J^2) = \text{Hom}_{\mathbb{C}}(\Phi^* \mathcal{F}_+, \mathcal{S}_{0,+})$$

on the intersection.  $\square$

We refer to the Morita isomorphism (10) as the canonical twisted  $\text{Spin}_c$ -structure of a  $q$ -Hamiltonian manifold.

*Remark 6.3.* In particular, we see that the third integral Stiefel-Whitney class of any q-Hamiltonian  $SU(2)$ -space satisfies

$$W^3(M) = 2\Phi^*x$$

where  $x \in H^3(SU(2), \mathbb{Z})$  is the generator. Since this is a 2-torsion class, it follows that  $4\Phi^*x = 0$ . The fact that  $\Phi^*x$  is torsion is a consequence of the condition  $d\omega = -\Phi^*\eta$ . The more precise statement relies on the minimal degeneracy condition  $\ker(\omega) \cap \ker(d\Phi) = 0$ .

## 7. PRE-QUANTIZATION OF Q-HAMILTONIAN $SU(2)$ -SPACES

Suppose  $(M, \omega, \Phi)$  is a q-Hamiltonian  $SU(2)$ -space. The conditions  $d\omega = -\Phi^*\eta$  and  $d\eta = 0$  mean that the pair  $(\omega, -\eta)$  defines a cocycle for the relative de Rham complex  $^4 \Omega^\bullet(\Phi)$ . For  $k > 0$ , we define a *level  $k$  pre-quantization* of  $(M, \omega, \Phi)$  to be a lift of the class  $k[(\omega, -\eta)] \in H^3(\Phi, \mathbb{R})$  to a class in  $H^3(\Phi, \mathbb{Z})$ .

*Remark 7.1.* One can similarly define an *equivariant level  $k$  pre-quantization* to be an integral lift of  $k[(\omega, -\eta_{SU(2)})] \in H^3_{SU(2)}(\Phi, \mathbb{R})$ . However, the equivariance is automatic: Indeed, for any simply connected compact Lie group  $G$ , and any  $G$ -space  $M$  one has  $H_G^p(M, \mathbb{Z}) = H^p(M, \mathbb{Z})$  for  $p \leq 2$ , and if  $\Phi: M \rightarrow G$  is an equivariant map one has  $H_G^p(\Phi, \mathbb{Z}) = H^p(\Phi, \mathbb{Z})$  for  $p \leq 3$ . See e.g. [30].

**Lemma 7.2.** *If  $(M, \omega, \Phi)$  admits a level  $k$  pre-quantization, then the set of such pre-quantizations is a principal homogeneous space under the group  $\text{Tor}(H^2(M, \mathbb{Z}))$  of flat line bundles over  $M$ .*

*Proof.* Clearly, the set of pre-quantizations is a principal homogeneous space under  $\text{Tor}(H^3(\Phi, \mathbb{Z}))$ . Since  $H^3(SU(2), \mathbb{Z}) = \mathbb{Z}$  has no torsion,  $\text{Tor}(H^3(\Phi, \mathbb{Z}))$  lies in the image of the map  $H^2(M, \mathbb{Z}) \rightarrow H^3(\Phi, \mathbb{Z})$  in the long exact sequence for relative cohomology. But this map is injective since  $H^2(SU(2), \mathbb{Z}) = 0$ , and hence restricts to an isomorphism of the torsion subgroups.  $\square$

The class  $k[(\omega, -\eta)]$  is integral if and only if it takes integer values on all relative 3-cycles: That is, for every smooth singular 2-cycle  $\Sigma \in C_2(M)$ , and every smooth singular 3-chain  $\Gamma \in C_3(SU(2))$  bounding  $\Phi(\Sigma)$ , we must have

$$(11) \quad k\left(\int_{\Gamma} \eta + \int_{\Sigma} \omega\right) \in \mathbb{Z}.$$

(Given  $\Sigma$ , it is actually enough, by the integrality of  $\eta$ , to check the condition for for *some*  $\Gamma$  bounding  $\Phi(\Sigma)$ .) If  $H^2(M, \mathbb{R}) = 0$ , there is a much simpler criterion [30]: Let  $x \in H^3(SU(2), \mathbb{Z})$  be the generator. Since  $\Phi^*[\eta] = 0$ , the class  $\Phi^*x$  is torsion. If  $H^2(M, \mathbb{R}) = 0$ , then  $(M, \omega, \Phi)$  is pre-quantizable at level  $k$  if and only if

$$(12) \quad k\Phi^*x = 0.$$

**Proposition 7.3.** *The conjugacy class  $\mathcal{C}$  of  $t \in T_{[0, \rho]} \subset SU(2)$  is pre-quantizable at level  $k$  if and only if  $t = \exp(\frac{n}{k}\rho)$  for some  $n \in \{0, 1, \dots, k\}$ .*

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<sup>4</sup>Recall that for any morphism of cochain complexes  $F^\bullet: C^\bullet \rightarrow \tilde{C}^\bullet$ , the relative cohomology  $H^\bullet(F)$  is the cohomology of the algebraic mapping cone  $(\tilde{C}^{k-1} \oplus C^k, d)$ , with differential  $d(x, y) = (F(y) - dx, dy)$ . In our case  $F = \Phi^*$ , acting on differential forms or on singular cochains, and we write  $H(\Phi, \cdot)$  for the relative cohomology.

*Proof.* It is enough to check Criterion (11) for  $\Sigma = \mathcal{C}$ . Write  $t = \exp(u\rho)$  with  $u \in [0, 1]$ . Let  $\mathcal{O}$  be the adjoint orbit of  $u\rho$ , so that  $\mathcal{C} = \Phi(\mathcal{O})$ . As above, let  $V \subset \mathfrak{su}(2)$  be the open ball of radius  $\frac{1}{\sqrt{2}}$ . Then  $\mathcal{O}$  is the boundary of  $V_u = uV$ , and we compute, with  $\Gamma = \Phi(\overline{V_u})$ ,

$$\int_{\Gamma} \eta = \int_{V_u} \exp^* \eta = \int_{V_u} d\varpi = \int_{\mathcal{O}} i_{\mathcal{O}}^* \varpi = \int_{\mathcal{O}} \omega_{\mathcal{O}} - \int_{\mathcal{C}} \omega_{\mathcal{C}}.$$

Hence

$$k \left( \int_{\Gamma} \eta + \int_{\mathcal{C}} \omega_{\mathcal{C}} \right) = k \int_{\mathcal{O}} \omega_{\mathcal{O}}$$

which is an integer if and only if the orbit through  $k u \rho$  is integral, i.e.  $k u \in \mathbb{Z}$ .  $\square$

**Proposition 7.4.** *The 4-sphere  $S^4$  and the double  $D(\mathrm{SU}(2))$  are pre-quantizable at any integer level  $k$ . More generally, this is the case for any  $q$ -Hamiltonian  $\mathrm{SU}(2)$ -space  $(M, \omega, \Phi)$  with vanishing second homology. The double  $D(\mathrm{SO}(3))$  (viewed as a  $q$ -Hamiltonian  $\mathrm{SU}(2)$ -space) is pre-quantizable at level  $k$  if and only if  $k$  is even.*

The condition for  $D(\mathrm{SO}(3))$  was first obtained by Derek Krepski [30].

*Proof.* In each of these examples we have  $H^2(M, \mathbb{R}) = 0$ , hence it suffices to find all  $k$  such that  $k\Phi^*x = 0$ . For  $M = S^4$ , one has  $\Phi^*x = 0$  since  $H^3(S^4, \mathbb{Z}) = 0$ . For  $M = D(\mathrm{SU}(2))$ , one again has  $\Phi^*x = 0$ , by the properties of  $x$  under group multiplication and inversion ( $\mathrm{Mult}^*x = \mathrm{pr}_1^*x + \mathrm{pr}_2^*x$ ,  $\mathrm{Inv}^*x = -x$ .) For  $M = D(\mathrm{SO}(3))$ , one checks that the torsion subgroup of  $H^3(M, \mathbb{Z})$  is  $\mathbb{Z}_2$ , so that  $M$  is pre-quantizable at either all levels or at all even levels. We claim that  $M$  is *not* pre-quantizable at level 1. To see this consider the symplectic submanifold  $T' \times T' \subset D(\mathrm{SO}(3))$ , where  $T'$  is the maximal torus in  $\mathrm{SO}(3)$  given as the image of  $T$ . For the symplectic volume one finds, (see 11.1 below)

$$\mathrm{vol}(T' \times T') = \frac{1}{4} \mathrm{vol}(T \times T) = \frac{2}{4} = \frac{1}{2}.$$

By Criterion (11), with  $\Sigma = T' \times T'$  and  $\Gamma = \emptyset$ , the pre-quantized levels  $k$  must satisfy  $k \int_{\Sigma} \omega \in \mathbb{Z}$ , hence they must be even.  $\square$

Finally, we remark that if  $(M_i, \omega_i, \Phi_i)$  are pre-quantized at level  $k$ , then their fusion product  $M_1 \times M_2$  inherits a pre-quantization at level  $k$ .

For an ordinary Hamiltonian  $\mathrm{SU}(2)$ -space  $(M, \omega_0, \Phi_0)$ , a pre-quantization is an integral lift of the class of the equivariant symplectic form. More generally, by a *level  $k$  pre-quantization* of such a space we mean a pre-quantization of  $(M, k\omega_0, k\Phi_0)$ . Geometrically, the lift is realized as the equivariant Chern class of an equivariant *pre-quantum line bundle* over  $M$ .

**Proposition 7.5.** *A level  $k$  pre-quantization of a  $q$ -Hamiltonian  $\mathrm{SU}(2)$ -space  $(M, \omega, \Phi)$  is equivalent to a pair of level  $k$  pre-quantizations of the Hamiltonian  $\mathrm{SU}(2)$ -spaces  $(M_{\pm}, \omega_{0,\pm}, \Phi_{0,+})$ , with the property that the pre-quantum line bundles  $L_{\pm} \rightarrow M_{\pm}$  satisfy*

$$L_- \cong L_+ \otimes \Phi^* J^k$$

on the overlap  $M_{\mathrm{reg}} = M_+ \cap M_-$ .

*Proof.* Let  $\Phi_{\pm}: M_{\pm} \rightarrow \mathrm{SU}(2)_{\pm}$  be the restrictions of  $\Phi$ . Since  $\mathrm{SU}(2)_{+}, \mathrm{SU}(2)_{-}$  retract onto  $e, c$  respectively, the long exact sequences in relative cohomology give isomorphisms  $H^2(M_{\pm}, \cdot) \xrightarrow{\cong} H^3(\Phi_{\pm}, \cdot)$ , and a commutative diagram,

$$\begin{array}{ccc} H^3(\Phi, \mathbb{Z}) & \longrightarrow & H^3(\Phi_{\pm}, \mathbb{Z}) \cong H^2(M_{\pm}, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^3(\Phi, \mathbb{R}) & \longrightarrow & H^3(\Phi_{\pm}, \mathbb{R}) \cong H^2(M_{\pm}, \mathbb{R}) \end{array}$$

The lower horizontal map is given on  $k[(\omega, -\eta)]$  by

$$k[(\omega, -\eta)] \mapsto k[\omega_{\pm} + \Phi_{\pm}^* \varpi_{\pm}] = k[\omega_{0,\pm}].$$

To give a parallel discussion of the upper horizontal map, let  $C^k(\cdot, R) = \mathrm{Hom}(C_k(\cdot), R)$  denote the complex of smooth singular cochains, with coefficient in the ring  $R$ . We have two natural cochain maps,

$$C^k(\cdot, \mathbb{Z}) \rightarrow C^k(\cdot, \mathbb{R}) \leftarrow \Omega^k(\cdot).$$

Let  $\eta^{\mathbb{Z}} \in C^3(\mathrm{SU}(2), \mathbb{Z})$  be a smooth singular cocycle whose image in  $C^3(\mathrm{SU}(2), \mathbb{R})$  is cohomologous to the image of  $\eta$ , and let  $\varpi_{\pm}^{\mathbb{Z}} \in C^2(\mathrm{SU}(2)_{\pm}, \mathbb{Z})$  be primitives of the restriction of  $\eta^{\mathbb{Z}}$  to  $\mathrm{SU}(2)_{\pm}$ . Let  $\sigma^{\mathbb{Z}} \in C^2(M, \mathbb{Z})$  be such that  $d\sigma^{\mathbb{Z}} = -k\Phi^*\eta^{\mathbb{Z}}$ , and such that  $[(\sigma^{\mathbb{Z}}, k\eta^{\mathbb{Z}})] \in H^3(\Phi, \mathbb{Z})$  represents the lift of  $k[(\omega, -\eta)]$  given by the pre-quantization. The upper map in the commutative diagram above is given on  $[(\sigma^{\mathbb{Z}}, k\eta^{\mathbb{Z}})]$  by

$$[(\sigma^{\mathbb{Z}}, k\eta^{\mathbb{Z}})] \mapsto [\sigma_{\pm}^{\mathbb{Z}} + k\Phi^* \varpi_{\pm}^{\mathbb{Z}}].$$

Hence  $[\sigma_{\pm}^{\mathbb{Z}} + k\Phi^* \varpi_{\pm}^{\mathbb{Z}}] \in H^2(M_{\pm}, \mathbb{Z})$  are integral lifts of  $k[\omega_{0,\pm}]$ . Let  $L_{\pm} \rightarrow M_{\pm}$  be the corresponding  $\mathrm{SU}(2)$ -equivariant pre-quantum line bundles, so that

$$c_1(L_{\pm}) = [\sigma_{\pm}^{\mathbb{Z}} + k\Phi^* \varpi_{\pm}^{\mathbb{Z}}].$$

On the overlap,  $M_{\mathrm{reg}} = M_{+} \cap M_{-}$ , the difference between the 2-cocycles  $\sigma_{\pm}^{\mathbb{Z}} + \Phi^* \varpi_{\pm}^{\mathbb{Z}}$  is  $k\Phi^*(\varpi_{-}^{\mathbb{Z}}|_{\mathrm{SU}(2)_{\mathrm{reg}}} - \varpi_{+}^{\mathbb{Z}}|_{\mathrm{SU}(2)_{\mathrm{reg}}})$ . The 2-cochain  $\varpi_{-}^{\mathbb{Z}}|_{\mathrm{SU}(2)_{\mathrm{reg}}} - \varpi_{+}^{\mathbb{Z}}|_{\mathrm{SU}(2)_{\mathrm{reg}}} \in C^2(\mathrm{SU}(2)_{\mathrm{reg}}, \mathbb{Z})$  is closed, and its cohomology class is an integral lift of  $[\varpi_{-}|_{\mathrm{SU}(2)_{\mathrm{reg}}} - \varpi_{+}|_{\mathrm{SU}(2)_{\mathrm{reg}}}] = \Psi^*[\omega_{\mathcal{O}}] \in H^2(\mathrm{SU}(2)_{\mathrm{reg}}, \mathbb{R})$ . Hence it represents the Chern class  $c_1(J)$ . We have shown that

$$c_1(L_{-}|_{M_{\mathrm{reg}}}) - c_1(L_{+}|_{M_{\mathrm{reg}}}) = k\Phi^* c_1(J)$$

and consequently  $L_{-}|_{M_{\mathrm{reg}}} \cong L_{+}|_{M_{\mathrm{reg}}} \otimes \Phi^* J^k$ . Conversely, given a pair of pre-quantum line bundles  $L_{\pm}$  with this property, we may retrace the steps of this proof to obtain an integral lift of  $[k(\omega, -\eta)]$ .  $\square$

In particular, we see that if  $(M, \omega, \Phi)$  is pre-quantized at level  $k$ , and  $e$  is a regular value of  $\Phi$ , then the symplectic quotient  $M//\mathrm{SU}(2)$  inherits a level  $k$  pre-quantization. The corresponding pre-quantum line bundle over  $M//\mathrm{SU}(2)$  is  $L_{+}//\mathrm{SU}(2) = L_{+}|_{\Phi^{-1}(e)}/\mathrm{SU}(2)$  is a pre-quantum line bundle.

The pre-quantization result may be expressed in terms of Morita trivializations:

**Proposition 7.6.** *A level  $k$  pre-quantization of a  $q$ -Hamiltonian  $\mathrm{SU}(2)$ -space  $(M, \omega, \Phi)$  gives rise to a Morita isomorphism,*

$$\mathbb{C} \simeq_{\mathcal{E}} \Phi^* \mathcal{A}^k.$$



*Proof.* Pick Morita trivializations  $\mathbb{C} \simeq_{\mathcal{F}_{\pm}} \mathcal{A}^k$  over  $SU(2)_{\pm}$ , with  $\mathcal{F}_{-} \cong \mathcal{F}_{+} \otimes J^k$  on the overlap. The pre-quantum line bundles  $L_{\pm} \rightarrow M_{\pm}$  defined by the level  $k$  pre-quantization satisfy  $L_{-} \cong L_{+} \otimes \Phi^* J^k$  on the overlap. Hence the Hilbert space bundles

$$\mathcal{E}_{\pm} := \text{Hom}_{\mathbb{C}}(L_{\pm}, \Phi^* \mathcal{F}_{\pm})$$

(where  $\text{Hom}_{\mathbb{C}}$  denotes continuous bundle homomorphisms) glue to give the desired Morita isomorphism.  $\square$

**Proposition 7.7.** *Suppose  $(M, \omega, \Phi)$  is a level  $k$  pre-quantized  $q$ -Hamiltonian  $SU(2)$ -space. Assume that  $e, c$  are regular values of  $\Phi$ . Then the imploded cross-section  $(X, \omega_X, \Phi_X)$  inherits a level  $k$  pre-quantization.*

*Proof.* Let  $(M_{\pm}, \omega_{0,\pm}, \Phi_{0,\pm})$  carry the corresponding pre-quantum line bundles with  $L_{-} = L_{+} \otimes \Phi^* J^k$  on the overlap. Since  $X_{\pm} = (M_{\pm} \times \mathbb{C}^2) // SU(2)$  are ordinary Hamiltonian quotients, we obtain pre-quantizations of the Hamiltonian  $T$ -spaces  $(X_{\pm}, \omega_X, \Phi_X)$ . The pre-quantum line bundles  $L_{X_{\pm}}$  satisfy  $L_{X_{\pm}}|_Y \cong L_{\pm}|_Y$ , hence

$$L_{X_{-}}|_Y = L_{X_{+}}|_Y \otimes \Phi_Y^* J^k = L_{X_{+}}|_Y \otimes \mathbb{C}_{k\rho}.$$

We conclude that  $L_{X_{+}}$  and  $L_{X_{-}} \otimes \mathbb{C}_{-k\rho}$  patch to define a global  $T$ -equivariant pre-quantum line bundle  $L_X \rightarrow X$ .  $\square$

## 8. QUANTIZATION OF Q-HAMILTONIAN SU(2)-SPACES

We are now in position to define the quantization of pre-quantized  $q$ -Hamiltonian  $SU(2)$ -spaces. We begin with a quick overview of the quantization of ordinary *Hamiltonian*  $G$ -spaces  $(M, \omega, \Phi)$ . Choose an invariant almost complex structure on  $M$ , compatible with the symplectic form. Such an almost complex structure is unique up to equivariant homotopy, and hence the isomorphism class of the resulting equivariant  $\text{Spin}_c$ -structure given by a  $G$ -equivariant spinor bundle  $\mathcal{S}$  is independent of this choice. We obtain a Morita isomorphism  $\mathbb{C}l(TM) \simeq_{\mathcal{S} \otimes p} \mathbb{C}$ . Given a pre-quantum line bundle  $L \rightarrow M$ , one can twist by  $L$  to obtain a new  $\text{Spin}_c$ -structure  $\mathcal{S} \otimes L^{-1}$ , hence a Morita isomorphism

$$\mathbb{C}l(TM) \simeq_{\mathcal{S} \otimes p \otimes L} \mathbb{C}.$$

This allows us to define a push-forward map relative to  $p: M \rightarrow \text{pt}$ ,

$$p_*: K_0^G(M, \mathbb{C}l(TM)) \rightarrow K_0^G(\text{pt}) = R(G),$$

and to set  $\mathcal{Q}(M) = p_*([M]) \in R(G)$ . (For  $G = \{e\}$ , this is just an integer.) Equivalently,  $\mathcal{Q}(M)$  may be viewed as the equivariant index of the  $\text{Spin}_c$ -Dirac operator for the  $\text{Spin}_c$ -structure  $\mathcal{S} \otimes L^{-1}$ . The quantization procedure for Hamiltonian  $G$ -spaces is compatible with products:

$$(13) \quad \mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2).$$

For any  $g \in G$ , the value of the equivariant index  $\mathcal{Q}(M)$  at  $g$  may be computed by Atiyah-Segal's localization theorem. On the other hand, one has the Guillemin-Sternberg *quantization commutes with reduction* property: Let  $\mathcal{Q}(M)^G \in \mathbb{Z}$  be the multiplicity with which the trivial representation occurs in  $\mathcal{Q}(M)$ . Then [31, 32]

$$\mathcal{Q}(M)^G = \mathcal{Q}(M//G).$$

Here the index  $\mathcal{Q}(M//G)$  is well-defined if 0 is a regular value of  $\Phi$  and the  $G$ -action on  $\Phi^{-1}(0)$  is free. If the action is only locally free, then  $M//G$  is an orbifold and

the quantization is defined by the index theorem for orbifolds. In the general case, if 0 is not a regular value and  $M//G$  is a singular space,  $\mathcal{Q}(M//G)$  may be defined by partial desingularization of the singular symplectic quotient [32].

Suppose now that  $(M, \omega, \Phi)$  is a compact q-Hamiltonian  $SU(2)$ -space, pre-quantized at level  $k$ . By combining the Morita isomorphisms  $\Phi^* \mathcal{A}^2 \simeq_S \mathbb{C}l(TM)$  from Proposition 6.2 and  $\mathbb{C} \simeq_{\mathcal{E}} \Phi^* \mathcal{A}^k$  from Proposition 7.6 we obtain a Morita isomorphism

$$\mathbb{C}l(TM) \simeq_{S^{op} \otimes \mathcal{E}} \Phi^* \mathcal{A}^{k+2}.$$

This defines a push-forward map in  $K$ -homology,

$$K_0^{SU(2)}(M, \mathbb{C}l(TM)) \rightarrow K_0^{SU(2)}(SU(2), \mathcal{A}^{k+2}) \cong R_k(SU(2)).$$

**Definition 8.1.** Let  $(M, \omega, \Phi)$  be a compact q-Hamiltonian  $SU(2)$ -space, pre-quantized at level  $k$ . We define the *quantization*  $\mathcal{Q}(M) \in R_k(SU(2))$  to be the push-forward of the  $K$ -homology fundamental class  $[M] \in K_0^{SU(2)}(M, \mathbb{C}l(TM))$ ,

$$\mathcal{Q}(M) = \Phi_*([M]).$$

The properties of this quantization procedure for q-Hamiltonian spaces are very similar to that for the Hamiltonian case: In particular, the analogue to the ‘quantization commutes with products’ property (13) holds, with the left hand side involving the fusion product of q-Hamiltonian spaces, and the right hand side the product in  $R_k(SU(2))$ . However, while (13) is rather obvious in the Hamiltonian theory, its q-Hamiltonian counterpart is a non-trivial fact (proved in [2]). In what follows, we will focus on ‘localization’ and ‘quantization commutes with reduction’ for q-Hamiltonian  $SU(2)$ -spaces.

## 9. LOCALIZATION

We had mentioned in 2.2 that any  $\tau \in R_k(SU(2))$  is determined by its values  $\tau(t)$  at elements  $t \in T_{k+2}^{\text{reg}}$ . For a level  $k$  pre-quantized q-Hamiltonian  $SU(2)$ -space  $(M, \omega, \Phi)$ , the number  $\mathcal{Q}(M)(t)$  may be computed by localization to the fixed point set  $M^t$  of  $t$ . By equivariance, and since  $t$  is regular, the moment map takes the fixed point set to the maximal torus  $T = SU(2)^t$ .

**Proposition 9.1.** *The restriction  $\mathcal{A}^{k+2}|_T$  admits a  $T_{k+2}$ -equivariant Morita trivialization,*

$$\mathbb{C} \simeq_{\mathcal{G}} \mathcal{A}^{k+2}|_T.$$

*This Morita trivialization is uniquely determined (up to equivalence) by requiring that  $\mathcal{G}|_e$  extends to an  $SU(2)$ -equivariant Morita trivialization of  $\mathcal{A}^{k+2}|_e$ .*

*Proof.* Choose  $SU(2)$ -equivariant Morita trivializations  $\mathbb{C} \simeq_{\mathcal{F}_{\pm}} \mathcal{A}^{k+2}|_{SU(2)_{\pm}}$  such that on the overlap,  $\mathcal{F}_{-} \cong \mathcal{F}_{+} \otimes J^{k+2}$ . Restrict to  $T$ -equivariant Morita trivializations over

$$T \cap SU(2)_{+} = T_{(-\rho, \rho)}, \quad T \cap SU(2)_{-} = T_{(0, 2\rho)}.$$

The intersection  $T_{(-\rho, \rho)} \cap T_{(0, 2\rho)}$  has two connected components,  $T_{(0, \rho)}$  and  $T_{(\rho, 2\rho)}$ . The restrictions of  $J^{k+2}$  to the two components are

$$\begin{aligned} J^{k+2}|_{T_{(0, \rho)}} &= T_{(0, \rho)} \times \mathbb{C}_{(k+2)\rho}, \\ J^{k+2}|_{T_{(\rho, 2\rho)}} &= T_{(\rho, 2\rho)} \times \mathbb{C}_{-(k+2)\rho}. \end{aligned}$$

Let

$$\mathcal{G}_{+} = \mathcal{F}_{+}|_{T_{(-\rho, \rho)}}, \quad \mathcal{G}_{-} = \mathcal{F}_{-}|_{T_{(0, 2\rho)}} \otimes \mathbb{C}_{(k+2)\rho}.$$

Then  $\mathcal{G}_- \cong \mathcal{G}_+$  over  $T_{(0,\rho)}$ , while  $\mathcal{G}_- = \mathcal{G}_+ \otimes \mathbb{C}_{2(k+2)\rho}$  over  $T_{(\rho,2\rho)}$ . But  $T_{k+2}$  is exactly the subgroup of  $T$  acting trivially on  $\mathbb{C}_{2(k+2)\rho}$ . That is, the bundles  $\mathcal{G}_\pm$  glue to define a  $T_{k+2}$ -equivariant Morita trivialization

$$\mathbb{C} \simeq_{\mathcal{G}} \mathcal{A}^{k+2}|_T.$$

By construction,  $\mathcal{G}|_e$  extends to the unique (up to equivalence)  $SU(2)$ -equivariant trivialization  $\mathcal{F}_+|_e$  of  $\mathcal{A}|_e$ . Any other  $T_{k+2}$ -equivariant Morita trivialization differs from  $\mathcal{G}$  by twist with a  $T_{k+2}$ -equivariant line bundle. Since  $\dim T = 1$  we have  $H_{T_{k+2}}^2(T) = H_{T_{k+2}}^2(\text{pt})$ , hence such a line bundle is detected by its restriction to  $e$ . Since only the trivial  $T_{k+2}$ -representation extends to an  $SU(2)$ -representation, the proof is complete.  $\square$

*Remark 9.2.* The last part of the proof relied on  $\dim T = 1$ . Indeed, the corresponding statement for higher rank groups is more tricky [2].

**Proposition 9.3.** *Suppose  $\Phi: M \rightarrow SU(2)$  is an equivariant map, and that we are given an equivariant Morita isomorphism  $\mathbb{C}l(TM) \simeq_{\mathcal{E}} \Phi^* \mathcal{A}^{k+2}$ . Then, for all regular elements  $t \in T \cap SU(2)_{\text{reg}}$ , and any component of the fixed point set  $F \subset M^t$ , the restriction  $TM|_F$  inherits a distinguished  $T_{k+2}$ -equivariant  $\text{Spin}_c$ -structure.*

*Proof.* By equivariance, and since  $t$  is regular,  $\Phi$  restricts to a map  $\Phi_F: F \rightarrow SU(2)^t = T$ . Hence we have  $T_{k+2}$ -equivariant Morita isomorphisms

$$\mathbb{C} \simeq_{\Phi^* \mathcal{G}} \Phi^*(\mathcal{A}^{k+2}|_T) \simeq_{\mathcal{E} \circ \Phi|_F} \mathbb{C}l(TM|_F).$$

But a Morita trivialization of a Clifford algebra bundle is equivalent to a  $\text{Spin}_c$ -structure.  $\square$

Let  $\mathcal{L}_F \rightarrow F$  be the  $\text{Spin}_c$ -line bundle associated to this  $\text{Spin}_c$ -structure on  $TM|_F$ .

*Remark 9.4.* The line bundle  $\mathcal{L}_F$  may be described as follows. From  $\mathbb{C}l(TM) \simeq_{\mathcal{E}} \Phi^* \mathcal{A}^{k+2}$  we obtain a Morita trivialization,

$$\mathbb{C} \simeq \mathbb{C}l(TM) \otimes \mathbb{C}l(TM) \simeq_{\mathcal{E} \otimes \mathcal{E}} \Phi^* \mathcal{A}^{2k+4}.$$

Over  $M_\pm$ , we have another Morita trivialization of  $\Phi^* \mathcal{A}^{2k+4}$  coming from the defining Morita trivializations of  $\mathcal{A}$  over  $U_\pm$ . The two Morita trivializations are related by line bundles  $\mathcal{L}_\pm \rightarrow M_\pm$ , with  $\mathcal{L}_- = \mathcal{L}_+ \otimes \Phi^* J^{-(2k+4)}$  on the overlap. The restriction of  $J^{2k+4}$  to  $T$  is  $T_{k+2}$ -equivariantly trivial, and  $\mathcal{L}_F$  is the  $T_{k+2}$ -equivariant line bundle obtained by gluing  $\mathcal{L}_\pm|_{F \cap M_\pm}$ .

Using Proposition 9.3 we see that even though  $M$  does not come with a  $\text{Spin}_c$ -structure, the fixed point contributions from the usual Atiyah-Segal-Singer theorem [9, 8, 10] are well-defined. Indeed one has,

**Theorem 9.5** (Localization). *Suppose  $(M, \omega, \Phi)$  is a compact  $q$ -Hamiltonian  $SU(2)$ -space, pre-quantized at level  $k$ . For all  $t \in T_{k+2}^{\text{reg}}$ , the number  $\mathcal{Q}(M)(t)$  is given as a sum of fixed point contributions,*

$$\mathcal{Q}(M)(t) = \sum_{F \subset M^t} \mathcal{Q}(\nu_F)(t),$$

where  $\mathcal{Q}(\nu_F)(t)$  is defined using the  $T_{k+2}$ -equivariant  $\text{Spin}_c$ -structure on  $TM|_F$ .

The proof of Theorem 9.5 is parallel to the proof of the localization formula in Atiyah-Segal [8]; details will be given in [2]. In the cohomological form of the index theorem, the fixed point contributions  $\mathcal{Q}(\nu_F)$  are given as integrals of certain characteristic classes over  $F$  (cf. [17, 5])

$$\mathcal{Q}(\nu_F)(t) = (\sigma(\mathcal{L}_F)(t))^{1/2} \int_F \frac{\widehat{A}(F) \exp(\frac{1}{2}c_1(\mathcal{L}_F))}{D_{\mathbb{R}}(\nu_F, t)}.$$

Here  $\widehat{A}(F)$  is the  $\widehat{A}$ -class, and  $D_{\mathbb{R}}(\nu_F, t)$  is given on the level of differential forms by

$$D_{\mathbb{R}}(\nu_F, t) = e^{\frac{i\pi}{4} \text{rank}_{\mathbb{R}}(\nu_F)} \det_{\mathbb{R}}^{1/2}(1 - t^{-1} e^{\frac{1}{2\pi} \text{curv}_{\mathbb{R}}(\nu_F)}),$$

with  $\text{curv}_{\mathbb{R}}(\nu_F) \in \Omega^2(F, \mathfrak{o}(\nu_F))$  the curvature form for an invariant Riemannian connection. The expression in parentheses lies in  $\Omega(F, \text{End}(\nu_F))$ , with zeroth order term the identity, and the (positive) square root of its determinant is well-defined. Finally  $\mathcal{L}_F$  is the line bundle associated to the  $\text{Spin}_c$ -structure on  $TM|_F$ , the phase factor  $\sigma(\mathcal{L}_F)(t) \in \text{U}(1)$  is given by the action of  $t$  on  $\mathcal{L}|_F$ , and  $\sigma(\mathcal{L}_F)(t)^{1/2}$  is a suitable choice of square root.<sup>5</sup> If  $F \subset M_+$ , the  $\text{Spin}_c$ -structure on  $TM|_F$  is defined by the almost complex structure on  $M_+$ , twisted by the line bundle  $L_+$ . Hence, the fixed point contribution can be written in ‘Riemann-Roch’ form:

$$\mathcal{Q}(\nu_F)(t) = \sigma(L_+|_F)(t) \int_F \frac{\text{Td}(F) \text{ch}(L_+|_F)}{D(\nu_{F,+}, t)},$$

where  $D(\nu_{F,+}, t)$  is the equivariant characteristic class

$$D(\nu_{F,+}, t) = \det_{\mathbb{C}}(1 - t^{-1} e^{\frac{i}{2\pi} \text{curv}_{\mathbb{C}}(\nu_{F,+})}),$$

with  $\text{curv}_{\mathbb{C}}(\nu_{F,+})$  the curvature form for an invariant Hermitian connection, and  $\sigma(L_+|_F)(t)$  the phase factor defined by the action of  $t$  on  $L_+|_F$ . There is a similar formula for the case  $F \subset M_-$ :

$$\mathcal{Q}(\nu_F)(t) = -t^{(k+2)\rho} \sigma(L_-|_F)(t) \int_F \frac{\text{Td}(F) \text{ch}(L_-|_F)}{D(\nu_{F,-}, t)}.$$

If  $t = j(q^s)$  with  $s = 1, \dots, k+1$ , we have

$$-t^{(k+2)\rho} = (-1)^{s-1}.$$

This sign factor may be traced back to our choice of Morita trivialization of  $\mathcal{A}^{k+2}|_T$ , which was chosen to be compatible with the  $\text{SU}(2)$ -equivariant Morita trivialization of  $\mathcal{A}^{k+2}|_e$  (rather than that of  $\mathcal{A}^{k+2}|_c$ ).

*Remark 9.6.* A detailed check of the equivalence of the ‘ $\text{Spin}_c$ ’ and ‘Riemann-Roch’ forms of the fixed point contribution may be found in [5, Section 2.3]. In general, it is quite possible that  $F$  is contained neither in  $M_+$  nor in  $M_-$ : this happens for instance for  $M = D(\text{SO}(3))$ , as discussed in the final Section of this paper.

<sup>5</sup>The square root is determined as follows. Let  $\mathcal{S}_x$  be the fiber of the spinor module at any given  $x \in F$ . Choose a  $T_{k+2}$ -invariant complex structure on  $T_x M$ , compatible with the orientation. Let  $c_1, \dots, c_{n/2} \in \text{U}(1)$  be the eigenvalues (with multiplicities) for the action of  $t$  on  $T_x M$ , and  $u \in \text{U}(1)$  the action of  $t$  on the line  $\text{Hom}_{\mathbb{C}1(T_x M)}(\wedge^{\mathbb{C}} T_x M, \mathcal{S}_x)$ . Then

$$\sigma(\mathcal{L}_F)(t)^{1/2} = u \prod_{c_r \neq 1} c_r^{1/2},$$

using the square roots of  $c_r \neq 1$  with positive imaginary part.

*Remark 9.7.* The right hand side of the localization formula appears in [5], as a ‘working definition’ of the quantization of a q-Hamiltonian space. However, in [5] it was not understood how to view this expression as the localization of an appropriate equivariant object on  $M$ .

## 10. QUANTIZATION COMMUTES WITH REDUCTION

Suppose  $(M, \omega, \Phi)$  is a compact q-Hamiltonian SU(2)-space, with a pre-quantization at level  $k$ . For each  $l = 0, \dots, k$ , let  $\mathcal{C}_l$  be the conjugacy class of the element  $\exp(\frac{l}{k}\rho)$ . If SU(2) acts freely (resp. locally freely) on  $\Phi^{-1}(\mathcal{C}_l)$ , then

$$M//_{\mathcal{C}_l} \text{SU}(2) = (M \times \mathcal{C}_l) // \text{SU}(2) \cong \Phi^{-1}(\mathcal{C}_l) / \text{SU}(2)$$

is a smooth symplectic manifold (resp. orbifold), with a level  $k$  pre-quantization from  $M$ . The Riemann-Roch numbers

$$\mathcal{Q}(M//_{\mathcal{C}_l} \text{SU}(2)) \in \mathbb{Z}$$

are thus defined. If SU(2) does not act locally freely, it is still possible to define the Riemann-Roch numbers using a partial desingularization, as in [32].

**Theorem 10.1** (q-Hamiltonian quantization commutes with reduction). *Let  $(M, \omega, \Phi)$  be a level  $k$  pre-quantized q-Hamiltonian SU(2)-manifold, and  $\mathcal{Q}(M) \in R_k(\text{SU}(2))$  its quantization. Let  $N(l) \in \mathbb{Z}$  be the multiplicity of  $\tau_l$  in  $\mathcal{Q}(M)$ . Then*

$$N(l) = \mathcal{Q}(M//_{\mathcal{C}_l} \text{SU}(2))$$

where the right hand side denotes the level  $k$  quantization of the symplectic quotient.

A general proof of this result, for arbitrary simply connected groups, can be found in [5]. Here we will present a much simpler approach for the rank 1 case. It is modeled after a similar proof for the Hamiltonian case [31, Appendix].

**Proposition 10.2.** *Let  $(M, \omega, \Phi)$  be a level  $k$  pre-quantized q-Hamiltonian SU(2)-space. Suppose SU(2) acts (locally) freely on  $\Phi^{-1}(e), \Phi^{-1}(c)$ , so that the imploded cross-section  $(X, \omega_X, \Phi_X)$  is a smooth Hamiltonian T-space, with a pre-quantization at level  $k$ . Let  $N_X(l)$ ,  $l \in \mathbb{Z}$  be the multiplicity function for the Hamiltonian T-space  $X$ , and  $N(l)$ ,  $0 \leq l \leq k$  that for the q-Hamiltonian SU(2)-space  $M$ . Then*

$$N_X(l) = \begin{cases} N(l) & \text{if } 0 \leq l \leq k \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* We will only consider the case that SU(2) acts freely on  $\Phi^{-1}(e), \Phi^{-1}(c)$ . The fact that  $N_X(l)$  vanishes unless  $0 \leq l \leq k$  is an easy special case of the Hamiltonian ‘quantization commutes with reduction’ theorem – see e.g. [18]. The statement is thus equivalent to showing that  $\mathcal{Q}(M)$  is the image, under the induction map  $R_k(T) \rightarrow R_k(\text{SU}(2))$ , of  $t^\rho \mathcal{Q}(M)(t) \in R(T)$  (restricted to  $T_{k+2}$ ). That is, we have to show that for all  $t = j(z)$ , with  $z \in \{q, q^2, \dots, q^{k+1}\}$ ,

$$\mathcal{Q}(M)(t) = \frac{t^\rho \mathcal{Q}(X)(t) - t^{-\rho} \mathcal{Q}(X)(t^{-1})}{t^\rho - t^{-\rho}} = \frac{\mathcal{Q}(X)(t)}{1 - t^{-2\rho}} + \frac{\mathcal{Q}(X)(t^{-1})}{1 - t^{2\rho}}.$$

The equivariant index theorem expresses  $\mathcal{Q}(M)(t)$  as a sum of fixed point contributions,  $\mathcal{Q}(\nu_F)(t)$ , as explained above. Since SU(2) acts freely on  $\Phi^{-1}(e), \Phi^{-1}(c)$ , the fixed point manifolds  $F$  are all contained in  $M_{\text{reg}}$ , hence we may work with the Riemann-Roch form of the fixed point contributions. By regularity,  $\Phi(F) \subset T^{\text{reg}}$ . Thus, either  $F \subset Y$ , or the image of  $F$  under the Weyl group action lies in  $Y$ . That

is, all fixed point manifolds come in pairs  $F, F'$ , with  $F \in Y$  and  $F'$  its image under the action of the non-trivial Weyl group element. We have,

$$\mathcal{Q}(\nu_{F'})(t) = \mathcal{Q}(\nu_F)(t^{-1}).$$

Now, since  $F \subset Y$  it also appears as a fixed point set in  $X$ . The normal bundle of  $F$  in  $M$  splits as a direct sum of its normal bundle  $\nu_F^X$  in  $X$  and the normal bundle of  $Y$  in  $M$ , the latter being  $T$ -equivariantly isomorphic to  $\mathbb{C}_\alpha = \mathbb{C}_{2\rho}$ . Hence, the fixed point contributions are related by

$$\mathcal{Q}(\nu_F)(t) = \frac{\mathcal{Q}(\nu_F^X)(t)}{1 - t^{-2\rho}}, \quad \mathcal{Q}(\nu_{F'})(t) = \frac{\mathcal{Q}(\nu_F^X)(t^{-1})}{1 - t^{2\rho}}.$$

Summing over all fixed point components  $F \subset Y^t$ , one obtains all contributions to the fixed point formula for  $X$ , *except* the contributions from  $F = M//\mathrm{SU}(2)$  and  $F = M//_c\mathrm{SU}(2)$ . From the explicit description of the normal bundle of  $M//\mathrm{SU}(2)$  as  $\Phi^{-1}(0) \times_{\mathrm{SU}(2)} \mathbb{C}^2$ , and the identity, for  $\xi \in \mathfrak{su}(2)$ ,

$$\det(1 - z^{-1}e^{-\xi}) = z^{-2} \det(1 - ze^\xi) = z^{-2} \det(1 - ze^{-\xi})$$

we obtain,

$$D(\nu_{M//\mathrm{SU}(2)}^X, z^{-1}) = z^{-2} D(\nu_{M//\mathrm{SU}(2)}^X, z).$$

Hence, the two terms for  $F = M//\mathrm{SU}(2)$  cancel in the fixed point formula for  $X$ . Similarly, the two contributions from  $F = M//_c\mathrm{SU}(2)$  cancel.  $\square$

*Proof of Theorem 10.1.* We have seen that  $N(l) = N_X(l)$ . From the ‘quantization commutes with reduction theorem’ for Hamiltonian  $\mathrm{U}(1)$ -spaces [18], we know that  $N_X(l)$  is the Riemann-Roch number of the level  $k$  quantization of a symplectic quotient of  $X$ :

$$N_X(l) = \mathcal{Q}(\Phi_X^{-1}(\frac{i\pi l}{k})/\mathrm{U}(1)) = \mathcal{Q}(M//_{c_l}\mathrm{SU}(2)). \quad \square$$

One obtains the multiplicities  $N(l)$  by the orthogonality relations (3). Writing  $N(l) = \mathcal{Q}(M//_{c_l}\mathrm{SU}(2))$  we obtain,

$$\mathcal{Q}(M//_{c_l}\mathrm{SU}(2)) = \sum_{s=1}^{k+1} \frac{|q^s - q^{-s}|^2}{2k+4} \pi(j(q^s)) \mathcal{Q}(M)(j(q^s)).$$

## 11. EXAMPLES

Using the localization formula, we can compute the quantizations  $\mathcal{Q}(M) \in R_k(\mathrm{SU}(2))$  for our basic examples. Recall that  $\tau_n$ ,  $n = 0, \dots, k$  are the basis elements of  $R_k(\mathrm{SU}(2))$ .

**11.1. The double.** We begin with the q-Hamiltonian  $\mathrm{SU}(2)$ -space  $D(\mathrm{SU}(2))$ . Recall that this space is pre-quantizable at any integer level  $k \geq 1$ .

**Proposition 11.1.** *The level  $k$  quantization of the double  $D(\mathrm{SU}(2))$  is given by*

$$\mathcal{Q}(D(\mathrm{SU}(2))) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (k+1-2j)\tau_{2j}.$$

Here  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ . Equivalently,

$$\mathcal{Q}(D(\mathrm{SU}(2)))(j(q^s)) = \frac{2k+4}{|q^s - q^{-s}|^2}$$

for  $s = 1, \dots, k+1$ .

*Proof.* We first verify the equivalence of the two formulas. Using the known formulas for products of  $\tau_n$ 's, one finds that

$$\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (k+1-2j) \tau_{2j} = \sum_{n=0}^k (\tau_n)^2.$$

Write  $z = q^s$ . Then

$$\begin{aligned} \sum_{n=0}^k (\tau_n(j(z)))^2 &= -\frac{1}{|z - z^{-1}|^2} \sum_{n=0}^k (z^{n+1} - z^{-(n+1)})^2 \\ &= \frac{1}{|z - z^{-1}|^2} \sum_{n=0}^k (2 - z^{2(n+1)} - z^{-2(n+1)}) = \frac{2k+4}{|z - z^{-1}|^2}, \end{aligned}$$

where the sum is evaluated as a geometric series (using  $z^{k+2} = (-1)^s$ ). We next compare this result to the fixed point computation for  $M = D(\mathrm{SU}(2))$  (the following computation may be found in [5]). Since the action of  $\mathrm{SU}(2)$  on  $M = \mathrm{SU}(2) \times \mathrm{SU}(2)$  is by conjugation on each factor, and  $j(z)$  is a regular element, its fixed point set is

$$M^{j(z)} = T \times T =: F.$$

Note that  $\Phi(F) = \{e\}$ , in particular  $F \subset M_+$ . The induced symplectic structure on  $F$  is the standard symplectic structure on  $T \times T$ , defined by the inner product:

$$\omega_F = \mathrm{pr}_1^* \theta_T \cdot \mathrm{pr}_2^* \theta_T$$

where  $\mathrm{pr}_i: T \times T \rightarrow T$  are the two projections. The symplectic volume of  $F$  is

$$\mathrm{vol}(F) = \int_{T \times T} \omega_F = \left( \int_T \theta_T \right) \cdot \left( \int_T \theta_T \right) = \alpha \cdot \alpha = 2.$$

The  $\mathrm{Spin}_c$ -line bundle  $\mathcal{L}_F$  comes from the level  $k+2$  Morita isomorphism  $\mathbb{C}l(TM) \simeq \Phi^* \mathcal{A}^{k+2}$ ,

$$\mathbb{C} \simeq \mathbb{C}l(TM) \otimes \mathbb{C}l(TM) \simeq \Phi^* \mathcal{A}^{2k+4}$$

hence it is isomorphic to the  $2k+4$ -th power of the level 1 pre-quantum line bundle over  $F$ . (We are using that  $H^2(M, \mathbb{Z}) = 0$ .) Hence  $\frac{1}{2}c_1(\mathcal{L}_F) = (k+2)\omega_F$ . By considering the action at  $x = (e, e) \in F$ , one checks that  $\zeta(\mathcal{L}_F)(t) = 1$ . Indeed, the  $\mathrm{Spin}_c$ -structure on  $T_x M$  extends to an  $\mathrm{SU}(2)$ -equivariant  $\mathrm{Spin}_c$ -structure, and the corresponding representation of  $\mathrm{SU}(2)$  on  $\mathcal{L}_F|_x$  is necessarily trivial. The normal bundle to  $F$  in  $M$  is a trivial bundle

$$\nu_F = \mathfrak{su}(2)/\mathfrak{t} \oplus \mathfrak{su}(2)/\mathfrak{t} = \mathbb{C} \oplus \mathbb{C}^-,$$

with  $T$  acting by weight 2 on the first summand and  $-2$  on the second summand. Hence

$$\frac{\zeta_F(t)^{1/2}}{D_{\mathbb{R}}(\nu_F, t)} = \frac{1}{|(1-z^2)(1-z^{-2})|} = \frac{1}{|z - z^{-1}|^2}.$$

Since finally  $\hat{A}(F) = 1$ , the fixed point contribution is

$$\chi(\nu_F, j(z)) = \int_F \frac{e^{(k+2)\omega_F}}{|z - z^{-1}|^2} = \frac{2k+4}{|z - z^{-1}|^2},$$

as claimed.  $\square$

Recall now that  $M(\Sigma_h) = D(\mathrm{SU}(2))^h // \mathrm{SU}(2)$  is the moduli space of flat  $\mathrm{SU}(2)$ -bundles over a surface of genus  $h$ . Using that quantization commutes with products, we have  $\mathcal{Q}(D(\mathrm{SU}(2))^h) = \mathcal{Q}(D(\mathrm{SU}(2)))^h$ . Together with the quantization commutes with reduction principle we hence obtain the Verlinde formula for this moduli space (cf. [40]):

$$\mathcal{Q}(M(\Sigma_h)) = \sum_{s=1}^{k+1} \left( \frac{|q^s - q^{-s}|^2}{2k+4} \right)^{1-h} = \sum_{s=1}^{k+1} \left( \frac{2 \sin^2(\frac{s\pi}{k+2})}{k+2} \right)^{1-h}.$$

**11.2. Conjugacy classes.** We had seen that the conjugacy classes  $\mathcal{C} \subset \mathrm{SU}(2)$  admitting a level  $k$  pre-quantizations are precisely those of elements  $\exp(\frac{n}{k}\rho)$  with  $0 \leq n \leq k$ .

**Proposition 11.2.** *The level  $k$  quantization of the conjugacy class  $\mathcal{C} = \mathrm{SU}(2) \cdot \exp(\frac{n}{k}\rho)$  is given by*

$$(14) \quad \mathcal{Q}(\mathcal{C}) = \tau_n.$$

*Equivalently, for  $s = 1, \dots, k+1$ ,*

$$(15) \quad \mathcal{Q}(\mathcal{C})(j(q^s)) = \frac{q^{s(n+1)} - q^{-s(n+1)}}{q^s - q^{-s}}.$$

*Proof.* The equivalence of the two formulations follows from the discussion in Section 2.2. Write  $z = q^s$ . If  $n < k$ , then  $\Phi(\mathcal{C}) \subset \mathrm{SU}(2)_+$ . The symplectic form on  $\mathcal{C} = \mathcal{C}_+$  identifies  $\mathcal{C}$  with the coadjoint orbit of  $\frac{n}{k}\rho$ , and the level  $k$  pre-quantization corresponds to the usual (level 1) pre-quantization of the orbit through  $n\rho$ . Written in Riemann-Roch form, the fixed point contributions for the conjugacy class are just the same as those for the coadjoint orbit, given by (15). If  $n = k$ , the conjugacy class  $\mathcal{C}$  coincides with the central element  $\{c\}$ . Since  $z^{k+2} = (-1)^s$  we have,

$$\chi_k(z) = \frac{z^{k+1} - z^{-(k+1)}}{z - z^{-1}} = \frac{z^{k+2} - z^{-(k+2)}z^2}{z^2 - 1} = -(-1)^s$$

which on the other hand is also the fixed point contribution for  $\mathcal{Q}(\mathcal{C})(j(z))$ , for  $\mathcal{C} \in \Phi^{-1}(\mathrm{SU}(2)_-)$ . This gives (15) for  $n = k$ .  $\square$

As a consequence, we may compute the level  $k$  quantization of

$$M(\Sigma_h^r; \mathcal{C}_1, \dots, \mathcal{C}_r) = D(\mathrm{SU}(2))^h \times \mathcal{C}_1 \times \dots \times \mathcal{C}_r$$

where  $\mathcal{C}_i$ ,  $i = 1, \dots, r$  are conjugacy classes of elements  $\exp(\frac{l_i}{k}\rho)$  with  $0 \leq l_i \leq k$ . One obtains,

$$\mathcal{Q}(M(\Sigma_h^r; \mathcal{C}_1, \dots, \mathcal{C}_r)) = \sum_{s=1}^{k+1} \left( \frac{|q^s - q^{-s}|^2}{2k+4} \right)^{1-h} \tau_{l_1}(q^s) \dots \tau_{l_r}(q^s).$$

For  $h = 0$  and  $r = 3$ , the right hand side of this formula are the fusion coefficients. That is,

$$\mathcal{Q}(M(\Sigma_0^3; \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)) = N_{l_1, l_2, l_3}^{(k)}.$$



**11.3. The 4-sphere.** Recall that the q-Hamiltonian space  $S^4$  admits a unique pre-quantization for all  $k$ .

**Proposition 11.3.** *The level  $k$  quantization of the 4-sphere is given by*

$$\mathcal{Q}(S^4) = \sum_{n=0}^k \tau_n.$$

Equivalently, for  $s = 1, \dots, k+1$

$$\mathcal{Q}(S^4)(j(q^s)) = \begin{cases} 2 |1 - q^{-s}|^{-2}, & s \text{ odd} \\ 0 & s \text{ even} \end{cases}$$

*Proof.* Write  $z = q^s$ . We first verify the equivalence of the two formulas:

$$\begin{aligned} \sum_{n=0}^k \tau_n(j(q(z))) &= \frac{1}{z - z^{-1}} \sum_{n=0}^k (z^{n+1} - z^{-(n+1)}) \\ &= \frac{1}{z - z^{-1}} \left( \frac{z - z^{k+2}}{1 - z} - \frac{z^{-1} - z^{-(k+2)}}{1 - z^{-1}} \right). \end{aligned}$$

If  $s$  is even, then  $z^{k+2} = 1$  and the two terms cancel. If  $s$  is odd, then  $z^{k+2} = -1$  and we obtain, writing  $(z - z^{-1}) = (1 - z^{-1})(z + 1)$ , that

$$\sum_{n=0}^k \tau_n(j(z)) = \frac{2}{(1 - z^{-1})(1 - z)} = \frac{2}{|1 - z^{-1}|^2}.$$

The fixed point set of  $t$  consists of the ‘north pole’  $\Phi^{-1}(e)$  and the ‘south pole’  $\Phi^{-1}(c)$ . By construction,  $S^4_{\pm}$  are identified with open balls in  $\mathbb{C}^2$ , with the standard SU(2)-action. Hence the weights for the  $T \subset \text{SU}(2)$ -action are  $+1, -1$  respectively, and the fixed point formulas give (using  $j(z)^{(k+2)\rho} = z^{k+2} = (-1)^s$ )

$$\mathcal{Q}(S^4)(j(z)) = \frac{1}{(1 - z)(1 - z^{-1})} - (-1)^s \frac{1}{(1 - z)(1 - z^{-1})},$$

as needed.  $\square$

**11.4. Moduli spaces of flat SO(3)-bundles.** The symplectic quotient

$$D(\text{SO}(3))^h // \text{SO}(3)$$

of an  $h$ -fold product of  $D(\text{SO}(3))$ ’s (viewed as  $q$ -Hamiltonian SO(3)-spaces) is the moduli space of flat SO(3)-bundles over a surface of genus  $h$ . It has two connected components, given as symplectic quotients of  $D(\text{SO}(3))^h$  where  $D(\text{SO}(3))$  is now viewed as a q-Hamiltonian SU(2)-space:

$$(16) \quad D(\text{SO}(3))^h // \text{SO}(3) = D(\text{SO}(3))^h // \text{SU}(2) \cup D(\text{SO}(3))^h //_c \text{SU}(2).$$

The two components correspond to the trivial and the non-trivial SO(3)-bundle over the surface. To obtain Verlinde numbers for these moduli spaces, we need to work out the quantization of the q-Hamiltonian SU(2)-space  $D(\text{SO}(3))$ .

We had seen that  $D(\text{SO}(3))$  is pre-quantizable at level  $k$  if and only if  $k$  is even. The different pre-quantizations are a principal homogeneous space under the torsion

subgroup of  $H^2(D(\mathrm{SO}(3)), \mathbb{Z})$ . In fact this group is all torsion, and

$$\begin{aligned} H^2(D(\mathrm{SO}(3)), \mathbb{Z}) &= H_{\mathbb{Z}_2 \times \mathbb{Z}_2}^2(D(\mathrm{SU}(2)), \mathbb{Z}) \\ &= H_{\mathbb{Z}_2 \times \mathbb{Z}_2}^2(\mathrm{pt}, \mathbb{Z}) \\ &= \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1)) \end{aligned}$$

Letting  $\mathbb{C}_\phi$  denote the 1-dimensional representation given by  $\phi \in \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$ , this group acts by tensoring with the flat line bundle

$$D(\mathrm{SU}(2)) \times_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mathbb{C}_\phi.$$

Let  $T' = T/\mathbb{Z}_2$  be the maximal torus in  $\mathrm{SO}(3)$ , and  $N(T) \subset \mathrm{SU}(2)$ ,  $N(T') \subset \mathrm{SO}(3)$  the normalizers. Similarly, for elements  $a, b, \dots$  of  $\mathrm{SU}(2)$  we denote by  $a', b', \dots$  their images in  $\mathrm{SO}(3)$ .

**Lemma 11.4.** *For any  $t \in T_{\mathrm{reg}} \subset \mathrm{SU}(2)$ , the fixed point set of its action on  $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$  is  $T' = T/\mathbb{Z}_2$  unless  $t^2 = c$ , in which case it is  $N(T') = N(T)/\mathbb{Z}_2$ .*

*Proof.* For  $a \in \mathrm{SU}(2)$ , the element  $a'$  is fixed under  $\mathrm{Ad}_t$  if and only if  $a$  is fixed up to a central element, i.e.  $tat^{-1}a^{-1} \in Z(\mathrm{SU}(2))$ . If this central element is  $e$ , this just means  $a \in T$ . If the central element is  $c$ , then  $at^{-1}a^{-1} = t^{-1}c$  shows that  $a \in N(T)$  represents the non-trivial Weyl element  $w$ , and  $c = tw(t^{-1}) = t^2$ . We have thus shown that the fixed point set of a regular element  $t$  is the image of  $T$  in  $\mathrm{SO}(3)$ , unless  $t^2 = c$  in which case it is the image of the normalizer  $N(T)$ .  $\square$

Let us consider the fixed contributions of any  $t = j(q^s)$ ,  $s = 1, 2, \dots, k+1$  for the q-Hamiltonian space  $D(\mathrm{SO}(3))$ , for  $k$  even. Note that  $t^2 = c \Leftrightarrow s = k/2 + 1$ , and so we have to consider two cases:

Case 1:  $s \neq 1 + \frac{k}{2}$ , i.e.  $t^2 \neq c$ . Then  $D(\mathrm{SO}(3))^t = T' \times T' =: F$  is connected, and its moment map image is  $\{e\}$ . Since  $\mathrm{SU}(2)$  acts trivially on the fiber of  $L_+$  at  $(e', e') \subset F$ , the action of  $t$  on  $L_+|_F$  is trivial. Hence the fixed point contribution is just 1/4 that of the corresponding fixed point manifold in  $D(\mathrm{SU}(2))$ :

$$\chi(\nu_F, t) = \frac{1}{4} \frac{2k+4}{|q^s - q^{-s}|^2} = \frac{1}{4 \sin^2(\frac{\pi s}{k+2})} \left( \frac{k}{2} + 1 \right).$$

Case 2:  $s = 1 + \frac{k}{2}$ , i.e.  $t^2 = c$  and  $q^s = i$ . Then  $D(\mathrm{SO}(3))^t = N(T') \times N(T')$  has four connected components, indexed by the elements of  $u = (u_1, u_2) \in W \times W = \mathbb{Z}_2 \times \mathbb{Z}_2$ . Choose

$$n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N(T)$$

as a lift of the non-trivial Weyl group element, and let  $n' \in N(T')$  its image. Then each fixed point component  $F_u$  has a base point

$$x_u \in \{(e', e'), (n', e'), (e', n'), (n', n')\}$$

with the property  $\Phi(x_u) = e$ . For any given choice of the pre-quantization, one finds that the contribution of the component labeled by  $u = (u_1, u_2)$  is of the form,

6

$$\chi(\nu_{F_u}, t) = \frac{\lambda(u)}{4} \frac{2k+4}{|q^s - q^{-s}|^2} = \frac{\lambda(u)}{4} \left(\frac{k}{2} + 1\right).$$

where  $\lambda(u) \in \mathbb{U}(1)$  is given by the action of  $t$  on  $L_+|_{m_u}$ . For  $u = (1, 1)$ , this phase factor is  $\lambda(u) = 1$  as above. The total fixed point contribution is obtained by summing over all  $u = (u_1, u_2)$ :

$$\mathcal{Q}(D(\mathrm{SO}(3)))(q^{k/2+1}) = \left(\frac{k}{2} + 1\right) \sum_u \frac{\lambda(u)}{4}.$$

Let  $\chi \in R_k(\mathrm{SU}(2))$  be defined by

$$(17) \quad \chi = \sum_{j=0}^{k/2} (-1)^j \tau_{2j} = \tau_0 - \tau_2 + \tau_4 \cdots + (-1)^{k/2} \tau_k.$$

Using the orthogonality relations for level  $k$  characters, one finds that

$$\chi(q^{k/2+1}) = \frac{k}{2} + 1, \quad \chi(q^s) = 0 \text{ for } s \neq k/2 + 1.$$

From the localization contributions, we see:

$$\mathcal{Q}(D(\mathrm{SO}(3))) = \frac{1}{4} \left( \mathcal{Q}(D(\mathrm{SU}(2))) + \sum_{u \neq (1,1)} \lambda(u) \chi \right).$$

It remains to understand the sum  $\sum_{u \neq (1,1)} \lambda(u)$ .

**Lemma 11.5.** *For every even  $k$ , and any  $\phi \in \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{U}(1))$ , the space  $D(\mathrm{SO}(3))$  admits a unique pre-quantization at level  $k$  with the property that*

$$\lambda(u) = (-1)^{k/2} \phi(u)$$

for all  $u \neq (1, 1)$ .

*Proof.* Changing the pre-quantization by  $\phi \in \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{U}(1))$  changes  $\lambda(u)$  to  $\tilde{\lambda}(u) = \lambda(u)\phi(u)$ . This shows uniqueness. For existence, we have to find a pre-quantization with  $\lambda(u) = (-1)^{k/2}$  for  $u \neq (1, 1)$ . In fact, it is enough to find such a pre-quantization for  $k = 2$ . (The general case will then follow by taking the  $k/2$ -th power of the pre-quantization at level 2.)

For  $k = 2$ , and any of the four possible pre-quantizations, write

$$\mathcal{Q}(D(\mathrm{SO}(3))) = \sum_{l=0}^2 N(l) \tau_l.$$

The localization formulas for  $q, q^2, q^3$  give equations

$$\begin{aligned} N(0) + \sqrt{2}N(1) + N(2) &= 1, \\ N(0) - N(2) &= \frac{1}{2} + \frac{1}{2} \sum_{u \neq (1,1)} \lambda(u), \\ N(0) - \sqrt{2}N(1) + N(2) &= 1. \end{aligned}$$

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<sup>6</sup>The computation is similar to that in Section 11.1. In particular, the symplectic volume of the 2-torus  $F_u$  may be computed by working out  $\omega_{F_u}$  in coordinates; one finds  $\mathrm{vol}(F_u) = 1/2$ . See [3] for more general calculations along these lines.

The first and third equation give  $N(1) = 0$  and  $N(0) + N(2) = 1$ . In particular,  $N(0) - N(2)$  is an odd integer. The second equation shows that  $\sum_{u \neq (1,1)} \lambda(u)$  is a real number. A change of pre-quantization produces a sign change of exactly two of the  $\lambda(u)$ 's with  $u \neq (1, 1)$ . Since  $\sum_{u \neq (1,1)} \tilde{\lambda}(u)$  is again a real number, it follows that all  $\lambda(u)$  are real, and hence equal to  $\pm 1$ . The number of  $\lambda(u)$ 's equal to  $-1$  must be odd, or else the second equation would give that  $N(0) + N(2) = 0$  or  $= 2$ , contradicting that  $N(0) - N(2)$  is odd. Hence, either all three  $\lambda(u)$ 's with  $u \neq (1, 1)$  are equal to  $-1$ , or exactly one of them  $\lambda(u)$  equals  $-1$  and the other two are equal to  $+1$ . The resulting four cases must correspond to the four pre-quantizations. In particular, there is a unique level 2 pre-quantization such that  $\lambda(u) = -1$  for all  $u \neq (-1, -1)$ .  $\square$

Let  $\delta_{\phi,1}$  be equal to 1 if  $\phi = 1$ , equal to 0 otherwise. Then  $\sum_u \phi(u) = 4\delta_{\phi,1}$ , i.e.  $\sum_{u \neq (1,1)} \phi(u) = -1 + 4\delta_{\phi,1}$ . It follows that

$$\mathcal{Q}(D(\mathrm{SO}(3))) = \frac{1}{4} \left( \mathcal{Q}(D(\mathrm{SU}(2))) + (-1)^{k/2} (-1 + 4\delta_{\phi,1}) \chi \right).$$

From the known expansions of  $\mathcal{Q}(D(\mathrm{SU}(2)))$  (Proposition 11.1) and  $\chi$  (Equation (17)) in the basis  $\tau_j$ , we finally obtain:

**Theorem 11.6.** *For  $k$  even, let  $D(\mathrm{SO}(3))$  carry the level  $k$  pre-quantization labeled by  $\phi \in \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$ . Then*

$$\mathcal{Q}(D(\mathrm{SO}(3))) = \frac{1}{4} \sum_{j=0}^{k/2} (k + 1 - 2j + (-1)^{j+k/2} (-1 + 4\delta_{\phi,1})) \tau_{2j}.$$

Equivalently, for  $s = 1, \dots, k + 1$ ,

$$\mathcal{Q}(D(\mathrm{SO}(3)))(j(q^s)) = \begin{cases} \frac{1}{4} \sin^{-2}(\frac{\pi s}{k+2}) (\frac{k}{2} + 1) & s \neq \frac{k}{2} + 1 \\ \frac{1}{4} (1 + (-1)^{k/2} (-1 + 4\delta_{\phi,1})) (\frac{k}{2} + 1) & s = \frac{k}{2} + 1 \end{cases}$$

Dividing into the various subcases, the formula reads,

$$\mathcal{Q}(D(\mathrm{SO}(3))) = \begin{cases} (\frac{k}{4} + 1)\tau_0 + (\frac{k}{4} - 1)\tau_2 + \frac{k}{4}\tau_4 + (\frac{k}{4} - 2)\tau_6 + \dots & \phi = 1, k \equiv 0 \pmod{4} \\ \frac{k}{4}\tau_0 + \frac{k}{4}\tau_2 + (\frac{k}{4} - 1)\tau_4 + (\frac{k}{4} - 1)\tau_6 + \dots & \phi \neq 1, k \equiv 0 \pmod{4} \\ (\frac{k-2}{4})\tau_0 + (\frac{k-2}{4} + 1)\tau_2 + (\frac{k-2}{4} - 1)\tau_4 + (\frac{k-2}{4})\tau_6 + \dots & \phi = 1, k \equiv 2 \pmod{4} \\ (\frac{k-2}{4} + 1)\tau_0 + (\frac{k-2}{4})\tau_2 + (\frac{k-2}{4})\tau_4 + (\frac{k-2}{4} - 1)\tau_6 + \dots & \phi \neq 1, k \equiv 2 \pmod{4} \end{cases}$$

Using this result, in combination with ‘quantization commutes with reduction’, it is now straightforward to compute the quantizations (Verlinde numbers) for the moduli spaces (16). Note that there are many different pre-quantizations, since one can choose a different  $\phi$  for each factor. The case with boundary (markings) is still more complicated, and will be discussed elsewhere.

*Remark 11.7.* For  $k \equiv 0 \pmod{4}$ , the result above was proved about eight years ago in joint work [3] with Anton Alekseev and Chris Woodward. Pantev [34] and Beauville [11] had earlier obtained similar results using techniques from algebraic geometry.

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